On qualitative theory of solutions to nonlinear partial differential equations.

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ON QUALITATIVE THEORY OF SOLUTIONS TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS.

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In this work I study certain aspects of qualitative behaviour of solutions to nonlinear PDEs. The thesis consists of introduction and three parts.

In the first part I study solutions of Emden-Fowler type elliptic equations in nondivergence form. In this part I establish the following results:

1. Asymptotic representation of solutions in conical domains;
2. A priori estimates for solutions to equations with weighted absorption term;
3. Existence and nonexistence of positive solutions to equations with source term in conical domains.

In the second part I study regularity properties of nonlinear degenerate parabolic equations. There are two results here:

- A Harnack inequality and the Hölder continuity for solutions of weighted degenerate parabolic equations with a time-independent weight from a suitable Muckenhoupt class;
- A new proof of the Hölder continuity of solutions.

The third part is propedeutic. In this part I gathered some facts and simple proofs relating to the Harnack inequality for elliptic equations. Both divergent and nondivergent case are considered. The material of this chapter is not new, but it is not very easy to find it in the literature. This chapter is built entirely upon the so-called "growth lemma" ideology (introduced by E.M. Landis).

DECLARATION

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

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STATEMENT 1

This thesis is the result of my own investigations, except where otherwise stated. Where correction services have been used, the extent and nature of the correction is clearly marked in a footnote(s).

Other sources are acknowledged by explicit references. A bibliography is appended.

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STATEMENT 2
I hereby give consent for my thesis, if accepted, to be available for photocopying and for inter-library loan, and for the title and summary to be made available to outside organizations.

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Contents

Chapter 1

Introduction

Some remarks on the general composition and layout of the thesis. The present thesis is built upon two groups of results, both coming from the qualitative theory of PDEs.

The first group of results, gathered in Chapter 3, concerns the so-called semi-linear non-divergent elliptic problems, the model case being

$$\triangle u = \pm |u|^{\sigma-1} u. \tag{1.0.1}$$

We study the superlinear case, which means that the constant $\sigma > 1$. There are three sections in this chapter. In Section 3.1 the a priori estimates for Emden-Fowler type inequalitites with absorption term are established. In Section 3.2 we study the question of existence and nonexistence of positive solutions to non-divergent semilinear equations in conical domains with the Dirichlet boundary condition. Section 3.3 is dedicated to characterization of asymptotic behavior of solutions to Emden-Fowler type equations in a special case.

The second group, gathered in Chapter 4, deals with the regularity properties of solutions of nonlinear degenerate (in the sense of E. DiBenedetto) parabolic equations, the classical examples given by the parabolic p-Laplace equation

$$u_t = \triangle_p u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right), \tag{1.0.2}$$

and the so-called 'Lions p-Laplacian',

$$u_t = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right). \tag{1.0.3}$$

The parameter p here is a constant, and I study the case when $p > 2$, which corresponds to the nonlinear 'degenerate' diffusion. It means that the 'conductivity coefficient' of the medium, which is $|\nabla u|^{p-2}$ for the p-Laplace, vanishes for small values of the gradient of a solution. Section 4.1 of this chapter deals with the so-called 'weighted' degenerate parabolic equations. Of concern in this section is the case when the 'conductivity coefficient' sufficiently depends on the position in space, i.e. behaves like $a(x)|\nabla u|^{p-2}$. In Section 4.2 we give an alternative proof of the Hölder continuity of solutions.

The choice of these two seemingly distant topics represents the evolution of the author's research interests. Moreover, as the reader will see, all of these results are very strongly influenced by the ideology of 'scaling', which is fundamental to the natural sciences.

This work consists of 5 papers, 3 of them published, 1 being currently in press and 1 submitted. One of this papers was written in collaboration with Irina Filimonova and one is a joint paper with Ugo Gianazza and Vincenzo Vespri.

All of these papers are essentially self-contained, each containing an introductory part and the relevant background material. Nevertheless, to make this work more coherent, I added a general introduction (Chapter 1) and a part containing some important results, fundamental for the topics I am concerned with (Chapter 2). Naturally, due to the different nature of the problems presented here, the introductory part and background material are divided into several parts. In Section 2.1 various facts related to the Sobolev spaces are gathered. In Section 2.2 I gathered many useful theorems from the theory of linear elliptic equations, these facts are used mainly in Chapter 3. Section 2.3 contains the more-or-less standard toolbox used in the theory of divergent parabolic equations.

The third part of the thesis (Chapter 5), discussing the 'growth lemma' and its applications, plays a linking, but also an aesthetical role. The results contained in it are not new, although it is not easy to find many of them in the easy-to-read form in the literature. Moreover, the 'growth lemma' ideology, which was due to E.M. Landis, sheds the brightest light on the classical regularity theory of PDEs. The achievement of E.M. Landis was the creation of the framework and the corresponding toolbox, which allows one to attack the problems like the Harnack inequality or regularity of solutions from the unified point of view. The arguments used by E.M. Landis have very simple geometric nature, which often allows one to easily change the underlying equation structure, for example, from non-divergent to divergent case - the core part of the technique remains untouched. In Section 5.1 I deal with equations of non-divergent structure. Section 5.2 is dedicated to equations of divergent structure.

Let us commence now the Emden-Fowler story.

Semilinear elliptic equations.

In the end of the 19th - beginning of 20th century the German physicist

Emden ([35, 36]) introduced the equations of the type (1.0.1) to study certain aspects of the behavior of the heated gas bodies ('Gaskügeln'). His model proved to be very useful for studying stellar dynamics, and its descendant, Lane-Emden-Ritter theory, is still used in astrophysics. The first models used only ordinary differential equations, like

$$\frac{d^2}{dt^2}u = \pm t^q u^\sigma, \tag{1.0.4}$$

and the solutions were supposed to be positive (due to their physical nature). A good reference source on the astrophysical applications of Emden's model and its derivatives are the classical books of Eddington [33] and Chandrasekhar [9, 10]. Emden's equation and the properties of its solutions soon attracted attention of the British mathematician Fowler, who thoroughly studied this subject ([41, 42, 43]) and obtained results concerning the asymptotic behavior of solutions. At approximately the same time, the very similar equation emerged in the nuclear physics under the name of the Thomas-Fermi equation, which is a special case of (1.0.1). After the works of Fowler, Emden's ordinary differential equation was continuously studied throughout the 20th century (for instance, [56, 101, 54, 55, 107]). The book of R. Bellman [7] contains the state-of-the-art (on the moment of publishing) survey of the results in asymptotic theory of the Emden-Fowler equations.

The ordinary differential equations of Emden-Fowler type and their generalizations, for instance, higher order equations like

$$\frac{d}{dx}\left(r_1(x)\frac{d}{dx}\left(r_2(x)\frac{d}{dx}\left(\dots\frac{d}{dx}\left(r_n(x)u\right)\right)\right)\right) = \pm Q(x)|u|^{\sigma-1}u \tag{1.0.5}$$

continue to pose many difficult interesting problems - see, for instance, the recent works of A. Kon'kov.

Later, equations of the type (1.0.1) found numerous applications in the wide range of areas of natural sciences: combustion theory ([100]), population dynamics and ecology models ([92, 96]), theory of pseudo-plastic fluids ([52, 83, 93]), to mention just a few.

In pure mathematics, equations of Emden-Fowler type have important applications in geometry, which was probably first explicitly noted by Osserman in [98]. The 'geometric' viewpoint was extensively used by W.M. Ni and others in [32, 58, 95].

I prefer looking at the solutions to equations of type (1.0.1) as at ground states of reaction-diffusion processes. Indeed, in equation (1.0.1) the Laplacian corresponds to the diffusion, while the nonlinear term $\pm|u|^{\sigma-1}u$ corresponds to some sort of reaction. Depending on the sign of the nonlinear term, we have either 'generating' reaction (for example, producing heat) in case of $-$, in which case we call this term the 'source term', or 'consuming' reaction ($+$ on the right-hand side of (1.0.1)), in which case the term is the 'absorption term'.

In my opinion, the two most striking features of equations of type (1.0.1) are the following. First, if one considers the absorption case, $\Delta u = |u|^{\sigma-1}u$, it is not hard to prove that the following ('Keller-Osserman') estimate holds: if u is a solution in the unit ball $\{|x| \le 1\}$, then $|u(0)| \le C$, where C is a constant independent of u. The physical interpretation is that the absorption is large for the large values of u, so it 'eats away' the high values of solutions. Among other consequences, the Keller-Osserman estimates guarantee the decay of solutions at infinity. Section 3.1 is dedicated to extending this result to a very general setting. The question answered in this section is the following. Put some weight depending on x in front of the nonlinear term. If this term is sufficiently small, the absorption effect diminishes. At some point, the absorption is no longer sufficient to guarantee the decay of solutions at infinity. If the weight has a power-like nature, the problem is easy to analyze, as long as the exponent of the power is not -2. My work in this direction began with obtaining estimates for solutions of $Lu = |x|^{-2}|u|^{\sigma-1}u$ with an elliptic operator L including the first-order term. Later I obtained the estimates for the general weight $Q(|x|)$ on the right-hand side. On the other hand, assume that the weight $Q(x)$ grows very rapidly at infinity. Then it turns out that the estimates which are valid for the mildly ('power-like') growing weights fail - the absorption can not 'catch up' with the growth of the coefficient, there is a delay, and a different formula should be used.

Second, for the equations with a 'source term', there is the following deep result of Gidas and Spruck [47]: Let u be a nonnegative solution of $\Delta u + u^\sigma = 0$ in $\mathbb{R}^n$ with $1 < \sigma < \frac{n+2}{n-2}$. Then $u \equiv 0$. From the physical point of view, this result says that the generation (say, of heat) is so powerful that even the whole space is unable to absorb and redistribute the heat. In the nonstationary case, we would speak about a 'blow-up' phenomenon in such a case. It is obvious that in many physical and geometric problems the absence of nontrivial solutions for certain values of parameters plays a critical role, whose value can hardly be overestimated. The result of Gidas and Spruck was extended in various directions, although the majority of papers (with the notable exception of [66]) deal either with the Laplacian or the divergent operator in the main part.

The non-divergent case presents its difficulties and rewards. Section 3.2 is an extended version of the joint paper with Irina Filimonova, in which we studied positive solutions to nondivergent semilinear equations in conical domains with the Dirichlet condition on the boundary. As opposed to the earlier works in this direction, we use the ideology of the 'Lemma on a Large Potential'. The idea of this lemma can be most easily exemplified by the following simple observation: Let u be a nonnegative solution of $\Delta u + CR^{-2}u \leq 0$ in the ball of radius R, $B_R = \{|x| < R\}$, where C is a positive constant. If $C > \lambda_1$, where λ_1 is the first eigenvalue of the Dirichlet Laplacian in the unit ball B_1, then $u \equiv 0$. Indeed, suppose that u is not identically zero. Then, using the strict maximum principle, we conclude that $u > 0$ in B_R. Moreover, we can assume that $R = 1$ since the coordinate transformation $x = Ry$ maps the ball B_R into the ball B_1 and the function $u_1(y) = u(Ry)$ satisfies the inequality $\Delta u_1 + Cu_1 \leq 0$ in B_1. Now, let ξ be an arbitrary function from $C_0^\infty(B_1)$. Multiply the inequality $\Delta u + Cu \leq 0$ by the test function $u^{-1}\xi^2$ and integrate by parts to obtain

$$\int_{B_1} \nabla u \cdot \nabla \left(\frac{\xi^2}{u}\right) dx \geq \int_{B_1} C\xi^2 \, dx.$$

Using the Leibnitz formula and the elementary inequality, we obtain

$$\begin{aligned}\int_{B_1} \left(\frac{|\nabla u|^2}{u^2} + C\right) \xi^2 \, dx &\leq \int_{B_1} 2\frac{\xi \nabla u}{u} \cdot \nabla \xi \, dx \\ &\leq \int_{B_1} \left(\frac{|\nabla u|^2}{u^2}\xi^2 + |\nabla \xi|^2\right) dx.\end{aligned}$$

Canceling the same terms on both sides, we obtain

$$C \int_{B_1} \xi^2 \, dx \leq \int_{B_1} |\nabla \xi|^2 \, dx.$$

If $C > \lambda_1$, the last inequality contradicts the definition of the first eigenvalue.

Thus the proof of the nonexistence can be reduced to proving that $u^\sigma(x) \geq C|x|^{-2}$ in any 'inner' subcone with sufficiently large C. This is achieved via the construction of explicit subsolutions and the use of the comparison principle. The construction of subsolutions and supersolutions (the latter are required in the 'existence' part of the proof) is where the main technicalities lie.

To illustrate the proof, I show here how it works for the simplest case. Consider a cone $\mathcal{K} \subset \mathbb{R}^n$ with a vertex at the origin. Let $\mathcal{K}_{a,b}$ denote $\mathcal{K} \cap \{a < |x| < b\}$. In $\mathcal{K}_{R,\infty}$ let u be a positive solution to the problem

$$\Delta u + u^{\sigma+1} = 0, \quad u = 0 \quad \text{on} \quad \partial\mathcal{K}.$$

Let α_- be a negative solution of the equation

$$\alpha^2 + (n-2)\alpha = \lambda_1,$$

where λ_1 is the first eigenvalue of $-\Delta_D$ on $\mathcal{K} \cap \{|x| = 1\}$. Denote the eigenfunction corresponding to λ_1 by ϕ. Obviously, for the function $w = r^{\alpha_-}\phi(\omega)$ we have $\Delta w = 0$. On the other hand, $\Delta u \leq 0$. Applying the Hopf-Oleinik lemma and the maximum principle, we obtain $u \geq cw$ with some positive constant c. Clearly, for any cone $\mathcal{K}' \Subset \mathcal{K}$ we have

$$u^\sigma \geq (cw)^\sigma \geq C_1 |x|^{\sigma\alpha_-} \geq C|x|^{-2}$$

if $\sigma < -\frac{2}{\alpha_-}$. We apply the Lemma on a Large Potential to prove that $u \equiv 0$.

On the other hand, for $\varepsilon > 0$ the function $w_1 = r^{\alpha_- - \varepsilon}\phi(\omega)$ solves the equation

$$\Delta w_1 = -c\phi(\omega) r^{\alpha_- - 2 - \varepsilon}, \quad \text{where} \quad c = \lambda_1 - (\alpha_- - \varepsilon)(\alpha_- + n - 2 - \varepsilon) > 0.$$

Hence, for sufficiently large $|x|$ we obtain

$$\Delta w_1 + w_1^{\sigma+1} = -c\phi r^{\alpha_- - 2 - \varepsilon} + (\phi)^{\sigma+1} r^{(\sigma+1)(\alpha_- - \varepsilon)} \leq 0$$

if $\sigma(\alpha_- - \varepsilon) < -2$, or $\sigma > \frac{-2}{\alpha_- - \varepsilon}$. Since ε is arbitrary, it follows that there exists a supersolution for any $\sigma > \frac{-2}{\alpha_-}$. Once a supersolution is in our hands, the proof of the existence result goes along the standard lines.

Another interesting feature of equations of type (1.0.1) is that in certain cases they can be studied as small perturbations of homogeneous (or, in critical cases, linear) equations. Naturally, this happens when a solution u is small. The effect reveals itself in its full power when one studies the asymptotic properties of solutions. In this case, there is a distinct borderline which separates two cases: in one case, the nonlinear term plays a principal role, and in the second case the nonlinear term is in some sense negligible. Indeed, the function $u = C|x|^{2/(1-\sigma)}$ with the constant $C = C(n, \sigma)$ is a solution to

$$\Delta u = |u|^{\sigma-1}u \quad \text{in} \quad \mathbb{R}^n \setminus \{0\} \tag{1.0.6}$$

for $\sigma \in (1, \frac{n}{n-2})$ and is a solution to

$$\Delta u = -|u|^{\sigma-1}u \quad \text{in} \quad \mathbb{R}^n \setminus \{0\} \tag{1.0.7}$$

for $\sigma > \frac{n}{n-2}$. In this special solution we see the full interaction between the diffusion and the reaction terms. On the other hand, one can easily verify (for example, using the results of [7] for ODEs), that in the exterior domain both equations (1.0.6) and (1.0.7) admit a solution with the asymptotics $u(x) = u(|x|) \sim c|x|^{2-n}$ for $\sigma > \frac{n}{n-2}$. The same argument readily shows that in the neighbourhood of the origin, solutions with asymptotics $u(x) \sim c|x|^{2-n}$ exist for both (1.0.6) and (1.0.7) for $1 < \sigma < \frac{n}{n-2}$. The basic intuition behind these facts is the following. Assume that $u(x) \sim |x|^\alpha$ in the neighbourhood of the origin. Then the right-hand side of (1.0.7) or (1.0.6) behaves like $|x|^{a\sigma}$. It is natural to expect that there is a solution w to the equation $Lw = \pm|u|^{\sigma-1}u$ which behaves like $C|x|^{a\sigma+2}$ (for clarification, I omit here all epsilons). The difference of u and w is a solution to the homogeneous equations. Moreover, if $a\sigma + 2 > a$, we see that $u - w$ has the same behavior as u. So, if we want to get the solution close to the fundamental solution, we should impose the condition $(2-n)\sigma + 2 > 2 - n$, which reads finally as $\sigma < \frac{n}{n-2}$. In the exterior domain these inequalities are reversed, i.e. we get the condition $(2-n)\sigma + 2 < 2 - n$, which yields $\sigma > \frac{n}{n-2}$. This line of arguments is developed in Section 3.3.

To complete this part of introduction, I would like to mention the excellent monograph [117] and more recent survey article [118]. They influenced me greatly during my work and provided the unceasing source of inspiration, useful references, techniques, and, above all, the possibility to look at this field of study from the bird's flight height, to perceive it as a whole. I refer the interested reader to these works for more information on the field and the extensive bibliography.

Now I go on to the next story.

Nonlinear degenerate parabolic equations.

Here I start by returning to the notion of diffusion. The equations which I study in the the first part of this thesis are *linear* in the main part, which means that the diffusion is comparable to the value of the gradient of a solution (and pointed in the opposite direction). The equations I study in the second part of the thesis are essentially nonlinear, the nonlinearity lying in the main, differential part of the equation. In physical models, equation (1.0.2) can describe, for example, the heat conduction, when heat flow's direction is still opposite to the direction of the gradient, but the heat conductivity of the medium depends on the value of the gradient of heat. In case $p > 2$ it has the following interpretation: if the difference in temperature between two neighbouring regions is large, then the heat flow between them is also very large, if this difference is small, the heat flow almost disappears. In the linear case the heat flow is proportional to the difference in the temperature level, the medium is 'indifferent' to the scale of solution, while in the degenerate nonlinear case the medium favours the 'high intensity' flows and penalizes the 'low-intensity' flows.

The p-Laplacian model in physics was used and developed by famous Soviet physicist G.I. Barenblatt. It is widely rumored that the works of Barenblatt and his discoveries played a colossal role in the Soviet H-bomb programme.

In the paper [6] G.I. Barenblatt studied the self-similar solutions of the parabolic p-Laplacian equations and made a beautiful discovery. Let us look at self-similar solutions to equation (1.0.2). Straightforward computation shows that they have the form

$$\mathcal{B}(x,t) = t^{-n/\lambda}\left\{C - \gamma_p\left(\frac{|x|}{t^{1/\lambda}}\right)^{\frac{p}{p-1}}\right\}_+^{\frac{p-1}{p-2}}, \quad t > 0, \tag{1.0.8}$$

where $\lambda = n(p-2)+p$ and

$$\gamma_p = \left(\frac{1}{\lambda}\right)^{\frac{1}{p-1}}\frac{p-2}{p}, \quad p > 2.$$

The function $\mathcal{B}$ solves the problem

$$\begin{gathered} u_t - \Delta_p u = 0, \quad \text{in} \quad \mathbb{R}^n \times (0,+\infty), \\ u(\cdot,0) = K\delta_0, \end{gathered}$$

where $K = K(n,p,C)$ and δ_0 stands for the Dirac mass concentrated at the origin. The first thing which one notices looking at formula (1.0.8) is that for any $t > 0$ the function $\mathcal{B}(x,t)$ has a finite support. More precisely, for each $t > 0$ the total mass is concentrated inside the ball $|x| \le ct^{1/\lambda}$, and outside this ball the solution is zero. Now, let us derive some other information on the qualitative behaviour of solutions from explicit formula (1.0.8). Take in this formula $t = t_0 = \rho^\lambda\left(\frac{\gamma_p}{C}\right)^{\frac{(p-1)\lambda}{p}}$, so that the support of the solution is the ball of radius ρ centered at the origin, and the maximum of $\mathcal{B}(x,t)$ in this ball is

$$M = C^{\frac{p-1}{p-2}}\rho^{-n}\left(\frac{\gamma_p}{C}\right)^{\frac{-n(p-1)}{p}} = \Lambda_0\rho^{-n}C^{\frac{(p-1)\lambda}{p(p-2)}}, \quad \Lambda_0 = \Lambda_0(n,p).$$

How much time does it take for the solution to cover the ball of radius 2ρ? It is easy to see that it happens at time

$$t_1 = (2\rho)^\lambda\left(\frac{\gamma_p}{C}\right)^{\frac{(p-1)\lambda}{p}}.$$

Easy calculation shows that the desired time difference (*waiting time* in the widely adopted terminology)

$$t_1 - t_0 = \Lambda_1 M^{2-p}\rho^p, \quad \Lambda_1 = \Lambda_1(n,p)$$

depends on both ρ and M. Therefore, the behavior of a solution is essentially regulated by its oscillation.

Another way to catch this dependence is via the 'scaling' method. Indeed, let us try to find the transformations $x = \rho y$, $u = kv$, $t = \theta\tau$, ρ, k and θ being positive constants, which preserve equation (1.0.2). It is easy to verify that these numbers should satisfy the relation $\theta = M^{2-p}\rho^p$.

The dependence of the propagation speed on the behavior of the solution implies the following consequence for the regularity theory: we can no longer efficiently work in the cylinders independent of solution, which was the case for the linear heat equation. If we try to write estimates in the cylinders, say, of the form $B_\rho \times (0, \rho^p)$ we will immediately face the lack of homogeneity, which makes these estimates virtually worthless. On the other hand, if we try to recover the 'proper' size of the parabolic cylinders by considering $|\nabla u|^{p-2}$ as the weight in the linear equation, we must be able to prove certain regularity of the gradient (say, prove that it belongs to Muckenhoupt-type classes). It could present some merit in simple cases like (1.0.2) or (1.0.3), but would be very hard to achieve, if achievable at all, for equations of a general structure. Moreover, the resulting estimates would depend on some integral characterizations of the gradient of a solution, which in the overwhelming majority of situations are not known *a priori.*

The seemingly inhomogeneous structure of the equation prevented it from being attacked by regularity people for about twenty years (the case $p = 2$ was more or less completely described in the monograph [81]). In 80's, E. DiBenedetto invented a brilliant simple trick which opened this field to ongoing research. His idea was to use parabolic cylinders of the form $Q = B_\rho \times (0, C\omega^{2-p}\rho^p)$, where ω is the oscillation of a solution in the cylinder Q. The essence of DiBenedetto's idea is that we can 'approximate' the gradient of a solution in Q by $\omega\rho^{-1}$. Equation (1.0.2) is thus 'approximated' by

$$u_t = div\left(\left(\frac{\omega}{\rho}\right)^{p-2} \nabla u\right), \tag{1.0.9}$$

which is a *linear* heat equation with the conductivity coefficient $(\omega/\rho)^{p-2}$. While the waiting time necessary for the solution of the heat equation $u_t = \triangle u$ to spread to the ball B_ρ^x is comparable to ρ^2, it is easy to see that the similar waiting time for equation (1.0.9) is

$$\rho^2 \Big/ \left(\frac{\omega}{\rho}\right)^{p-2} = \omega^{2-p}\rho^p.$$

In the geometry induced by these *intrinsic* cylinders, a solution to the p-Laplace equation behaves like a solution to the heat equation. Moreover, thus we recover the homogeneity of the corresponding integral estimates (of Caccioppolli type), which allows for the use of DeGiorgi-Ladyzhenskaja-Solonnikov-Uraltseva type arguments. However, the price for this is that the size of the cylinders we work with, now depends on the oscillation of a solution in this very cylinder, which is seemingly a vicious circle. This difficulty has to be overcome each time with different types of arguments.

The technique introduced by DiBenedetto is widely used now, for the p-Laplacian, the porous medium equations, equations with double nonlinearity, etc. The results for nonlinear equations are formulated in the intrinsic geometry. For example, the Harnack inequality, one of the most hard and deep results of the regularity theory of elliptic and parabolic equations, for the solutions of (1.0.3) has the following form:

Let u be a nonnegative solution of equation (1.0.2) in the cylinder $Q = \{|x| \leq \rho\} \times (-k^{2-p}\rho^p, \Lambda k^{2-p}\rho^p)$, where $\Lambda = \Lambda(n,p) > 0$. Let $u(0,0) \geq k$. Then $u(x, \Lambda k^{2-p}\rho^p) \geq \gamma k$ for $|x| < \frac{\rho}{2}$ with the positive constant $\gamma = \gamma(n,p)$. Two obvious differences between the Harnack inequality in the nonlinear degenerate parabolic setting and the Harnack inequality in the linear case are:
1) It takes some positive time for the solution to expand the positivity over the ball. In fact, using the example of the Barenblatt solution, one can easily see, that if the constant Λ is taken sufficiently small, then the infimum of $u(x, \Lambda k^{2-p}\rho^p)$ in the ball $B_{\rho/2}$ is zero. Which is quite unlike the linear case, where the speed of propagation is infinite, so $\inf_{B_{\rho/2}} u(x,t) > \gamma(t)k$, where $\gamma(t)$ is a function positive for all positive t, albeit going to zero as $t \to 0$.
2) The time required to expand the controlled positivity to the ball $B_{\rho/2}$ depends on the value of a solution at the initial point.

In this thesis I provide two results in this direction. The first one is the Harnack inequality for weighted degenerate parabolic equations, the example being

$$u_t = div(A(x)|\nabla u|^{p-2}\nabla u), \tag{1.0.10}$$

with the function $A(x)$ from the Muckenhoupt class $A_{1+p/n}$ — the exact definition and basic properties of the Muckenhoupt classes are given in the corresponding section. The function $A(x)$ is generally neither uniformly bounded from below nor uniformly bounded from above. It turns out, that the time needed for a solution u to expand its positivity to the ball of the radius $\rho/2$ now depends on a certain integral characteristic of the function A in the ball of the the radius ρ. This integral characteristic essentially depends on the point at which this ball is centered.

To illuminate this issue, assume that u is a solution to equation (1.0.10) in the cylinder $Q = B^x_\rho \times (0, k^{2-p}H(x,\rho))$, where $H(x,\rho)$ will be defined later. Moreover, assume that $u = 0$ on the lateral boundary of Q. Let us multiply equation (1.0.10) by u, integrate the result over Q, and do the integration in the term on the left-hand side and integration by parts in the integral on the right-hand side. Denote $T = k^{2-p}H(x,\rho)$. We obtain

$$\frac{1}{2}\int_{B^x_\rho} \left[u^2(y,T) - u^2(y,0)\right] dy = \int_Q A(y)|\nabla u|^p dydt. \tag{1.0.11}$$

The scaling factor k^{2-p} absorbs the inhomogeneity in u between the right-hand and left-hand sides (we choose k comparable with u). Next, we apply the dimensional analysis, to conclude that the following relation should hold if we want (1.0.11) to make physical sense:

$$|B_\rho^x| \sim \rho^{-p} H(x,\rho) \int_{B_\rho^x} A(y)dy.$$

Hence, we arrive at

$$H(x,\rho) \sim \rho^{n+p} \left(\int_{B_\rho^x} A(y)dy \right)^{-1}.$$

This function replaces ρ^p in all subsequent arguments. One would naturally expect that since now we work in the cylinders which depend on the point they are centered at, the modulus of continuity of a solution also varies from point to point. It is indeed so, but the Muckenhoupt classes proved to be 'good' enough to guarantee the uniform Hölder continuity of solutions, with the Hölder exponent independent of the point. If a careful reader compares the function $H(x,\rho)$ given here with the function $h(x,\rho)$ I introduce in Section 4.2 (and which plays the same role), he will notice that they are seemingly different. In the notation I use here,

$$h(x,\rho) = \left(\int_{B_\rho^x} (A(y))^{-n/p} dy \right)^{p/n}.$$

In fact, the Muckenhoupt condition says that for all $x \in \mathbb{R}^n$ and $\rho > 0$ we have

$$C_1 \le \frac{h(x,\rho)}{H(x,\rho)} \le C_2$$

with some positive constants C_1 and C_2.

Another problem I study here is the Hölder continuity of solutions to nonlinear parabolic equation of type (1.0.2). The result presented here is the product of collaboration with Ugo Gianazza and Vincenzo Vespri. This proof is much more transparent than the classical proof of DiBenedetto and uses the same ideas that led to the proof of the Harnack inequality in [28]. For those who are familiar with the latter work, the proof can be outlined by the next formula:

$$\text{Hölder} = \frac{1}{2}\text{Harnack}.$$

More precisely, the task, as usual, is to obtain the reduction of the oscillation of a solution in the smaller cylinder. As opposed to the linear case, the measure of the decrease in size of the subsequent cylinder is quite substantial (compare with the waiting time for the Harnack). Suppose that we work with the simplest equation, (1.0.2). The transformation $u = \beta v + c$, $x = \rho y$, $t = \beta^{2-p}\rho^p$ maps a solution u to the solution v of the same equation. Thus, we can safely assume that we work in the cylinder $Q_1 = B_8 \times (-2, L)$, where the constant L is to be chosen later, and in this cylinder $\max u = 1$, $\min u = 0$. To prove the reduction of oscillation, it is sufficient to prove the following:

ST1: If

$$|\{(x,t) \in B_1 \times (-1,0) \,:\, u(x,t) \geq \frac{1}{2}\}| \geq \frac{1}{2}$$

then there exists a positive constant γ such that

$$u(x,t) \geq \gamma \quad \text{in the cylinder} \quad Q_1 = B_1 \times (A, L)$$

with some positive constant A.

If the opposite holds, namely

$$|\{(x,t) \in B_1 \times (-1,0) \,:\, u(x,t) \geq \frac{1}{2}\}| \leq \frac{1}{2}$$

we can consider $u_1 = 1 - u$ and repeat all arguments for u_1.

The proof of **ST1** consists of the following steps:

I. We use the 'concentration of positivity' argument of [28] to obtain a cylinder

$$Q_3 = B^{x'}_{2\tau} \times (t' - C_1 4^{p-2}(2\tau)^p, t') \subset B_1 \times (-1,0)$$

such that

$$|\{(x,t) \in Q_3 : u(x,t) > \frac{1}{4}\}| > (1-\varepsilon)|Q'|$$

with sufficiently a small $\varepsilon > 0$. Here the constant $\tau = \tau(n,p)$.

II. Using the standard DiGiorgi-type lemma, we obtain

$$u(x,t) \geq \frac{1}{8} \quad \text{in} \quad Q_4 = B^{x'}_{\tau} \times (t' - C_2 8^{p-2}\tau^p, t').$$

III. For $x \in B^{x'}_{\tau/2}$ and $t \geq t'$ we obtain the estimate

$$u(x,t) \geq \psi(t) := \frac{1}{16}\left(1 + C_3 \frac{t-t'}{8^{p-2}\tau^p}\right)^{\frac{1}{2-p}}.$$

IV. Following [28], we perform the change of variables

$$u(x,t) = \psi(t)v(x,t), \tau = \tau(t)$$

where τ solves the problem

$$\frac{d\tau}{dt} = \psi^{p-2}(t), \quad \tau(t') = 0.$$

In the new variables, the function v is a supersolution of equation (1.0.2). Moreover, it possesses the 'heated core', which means that $v(x,t) \geq 1$ for all $x \in B^{x'}_{\tau/2}$. Applying another De Giorgi-type argument ('telescopic'), we find a constant γ and a cylinder $Q_5 = B^0_1 \times \big((L_1 - 2A_1)\gamma^{2-p}, L_1\gamma^{2-p}\big)$ with sufficiently large L_1 and $A_1 > 0$ such that

$$|\{(x,\tau) \in Q_5 : v(x,\tau) \leq \gamma\}| \leq \varepsilon |Q_5|$$

for a sufficiently small $\varepsilon > 0$. Now, the De Giorgi lemma implies that

$$v(x,\tau) \geq \gamma/2 \quad \text{in} \quad B^0_1 \times \big((L_1 - A_1), L_1\gamma^{2-p}\big).$$

Returning to the original variables, we complete the proof.

Chapter 2

Background material

This chapter contains the toolbox used in this thesis as well as some relevant classical results. I have taken them mainly from monograph [48], which is the accepted reference source for the modern theory of elliptic equations . The results on the Sobolev spaces can be also found in [1], [84] and many other books. The book of Maz'ja uses an approach quite different from the one introduced by Sobolev and subsequently used by many authors, while the book of Adams is written in more traditional style. For those interested in the classical theory of elliptic equations, I can suggest the monograph by K.Miranda [86]. It is a little bit outdated and does not contain any information on "weak" or "strong" solutions, but is at the same time the treasure chest of results related to classical solutions.

2.1 Sobolev Spaces and Imbedding Theorems.

I start with formulating the classical Sobolev imbedding theorem.

Theorem 2.1.1. *Let Ω be a Lipshitz domain on $\mathbb{R}^n$.*

(i) If $kp < n$, the space $W^{k,p}(\Omega)$ is continuously imbedded in $L^{p^*}(\Omega)$, $p^* = \frac{np}{n-kp}$, and compactly imbedded in $L^q(\Omega)$ for any $q < p^*$.

(ii) If $0 \leq m < k - \frac{n}{p} < m+1$, the space $W^{k,p}(\Omega)$is continuously imbedded in $C^{m,\alpha}(\bar{\Omega})$ with $\alpha = k - \frac{n}{p} - m$, and compactly imbedded in $C^{m,\beta}(\Omega)$ for any $\beta < \alpha$.

For an arbitrary domain Ω the same results hold if we replace $W^{k,p}(\Omega)$ with $W_0^{k,p}(\Omega)$.

The borderline case, $n = kp$, is described by the Yudovich-Pokhozhaev-Moser-Trudinger inequality. I refer the reader to the monograph [1] for detailed information on this inequality and related topics.

Theorem 2.1.2. *There exist positive constants c_1 and c_2 depending only on n*

and k such that if $u \in W_0^{k,p}(\Omega)$ with $n = kp$, then

$$\int_\Omega \exp\left(\frac{|u|}{c_1\|D^k u\|_{p;\Omega}}\right)^{\frac{p}{p-1}} dx \le c_2|\Omega|.$$

Quite often, the following interpolation inequality is useful.

Theorem 2.1.3. *Let $u \in W_0^{k,p}(\Omega)$. Then for any $\varepsilon > 0$ and $0 < |\beta| < k$,*

$$\|D^\beta u\|_{p;\Omega} \le \varepsilon\|u\|_{k,p;\Omega} + C\varepsilon^{\frac{|\beta|}{|\beta|-k}}\|u\|_{p;\Omega},$$

where $C = C(k)$. If Ω is a $C^{1,1}$ domain, the same result holds for $u \in W^{k,p}(\Omega)$, with the constant $C = C(k,\Omega)$.

It is useful to know the following extension and density results results.

Theorem 2.1.4. *Let $k \ge 1$ Ω be a $C^{k-1,1}$ domain in $\mathbb{R}^n$. Then for any open set $\Omega' \Supset \Omega$ there exists a bounded linear extension operator E from $W^{k,p}(\Omega)$ into $W_0^{k,p}(\Omega')$ such that $Eu = u$ in Ω and*

$$\|Eu\|_{k,p;\Omega'} \le C\|u\|_{k,p;\Omega} \quad \textit{for all} \quad u \in W^{k,p}(\Omega)$$

where $C = C(k,\Omega,\Omega')$.

Theorem 2.1.5. *The subspace $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$ for an arbitrary Ω. If Ω be a Lipshitz domain in $\mathbb{R}^n$ then $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$.*

It is often convenient to have the standard Sobolev-style representation formulae handy.

Theorem 2.1.6. *Let $u \in W_0^{1,1}(\Omega)$. Then*

$$u(x) = \frac{1}{n|B_1|}\int_\Omega \sum_{i=1}^n \frac{(x_i - y_i)D_i u(y)}{|x-y|^n}\,dy \quad a.e.\ in \quad \Omega.$$

Theorem 2.1.7. *Let Ω be convex and $u \in W^{1,1}(\Omega)$. Then*

$$|u(x) - u_S| \le \frac{d^n}{n|S|}\int_\Omega \frac{|x-y|\cdot|Du(y)|}{|x-y|^n}\,dy \quad a.e.\ in \quad \Omega$$

where

$$u_S = \frac{1}{|S|}\int_S u(x)dx, \quad d = diam\,\Omega,$$

and S is any measurable subset of Ω with $0 < |S| < \infty$.

To work with the equations of a divergent structure we need level cuts of Sobolev functions. Their basic properties are derived from the following two lemmas.

Theorem 2.1.8. *Let $u \in W^{1,1}(\Omega)$. Then $Du = 0$ a.e. on any set where u is constant.*

Theorem 2.1.9. *Let f be a piecewise smooth function on $\mathbb{R}$ with $f' \in L^\infty(\mathbb{R})$. Then if $u \in W^{1,1}(\Omega)$, we have $f \circ u \in W^{1,1}(\Omega)$. Furthermore, letting L denote the set of corner points of f, we have*

$$D(f \circ u) = \begin{cases} f'(u)Du & \text{if} \quad u \notin L, \\ 0 & \text{if} \quad u \in L. \end{cases}$$

2.2 Linear Elliptic Theory, Strong Solutions

This section is an extract of Chapter 9 ('Strong Solutions') of the classical book [48].

A function u is called a strong solution to the equation

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_j} + c(x)u = f \tag{2.2.1}$$

in a domain $\Omega \subset \mathbb{R}^n$ if u is a twice weakly differentiable function satisfying (2.2.1) almost everywhere (a.e.) in Ω. A function u is called a subsolution (supersolution) of equation (2.2.1) if u is twice weakly differentiable and satisfies $Lu \geq f$ ($Lu \leq f$) a.e. in Ω. The operator L in (2.2.1) is called elliptic in the domain Ω if the coefficient matrix $\mathcal{A} = \{a_{ij}\}$ is positive everywhere in Ω. Denote

$$\mathcal{D} = \det \mathcal{A}, \quad \mathcal{D}^* = \mathcal{D}^{1/n},$$
$$\lambda = \inf_{x \in \Omega, |\xi|=1} (A\xi, \xi), \quad \Lambda = \sup_{x \in \Omega, |\xi|=1} (A\xi, \xi).$$

The operator L is called strictly elliptic if $\lambda > 0$.

2.2.1 L^p estimates.

The next theorem is the celebrated Calderon-Zygmund inequality.

Theorem 2.2.1. *Let $f \in L^p(\Omega)$, $1 < p < \infty$, and let w be the Newtonian potential of f. Then $w \in W^{2,p}(\Omega)$, $\Delta w = f$ a.e. and*

$$\|D^2 w\|_p \leq C \|f\|_p \quad \text{where} \quad C = C(n,p).$$

The L^p estimates for solutions of Poisson's equation follow immediately.

Theorem 2.2.2. *Let Ω be a domain in $\mathbb{R}^n$, $u \in W_0^{2,p}(\Omega)$, $1 < p < \infty$. Then*

$$\|D^2 u\|_p \leq C \|\Delta u\|_p \quad \text{where} \quad C = C(n,p).$$

The technique of perturbation from the constant coefficients is used to carry over the Calderon-Zygmund estimate to equations with variable coefficients.

Theorem 2.2.3. *Let Ω be an open set in $\mathbb{R}^n$ and $u \in W^{2,p}_{loc}(\Omega) \cap L^p(\Omega)$, $1 < p < \infty$, be a strong solution of the equation $Lu = f$ in Ω where the coefficients of L satisfy, for positive constants λ and Λ,*

$$\begin{gathered} a_{ij} \in C(\Omega), \quad b_i,\ c \in L^\infty(\Omega), \quad f \in L^p(\Omega); \\ a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n; \\ |a_{ij}|,\ |b_i|,\ |c| \leq \Lambda, \end{gathered} \tag{2.2.2}$$

where $i, j = 1, \ldots, n$. Then for any domain $\Omega' \Subset \Omega$,

$$\|u\|_{2,p;\Omega'} \leq C\left(\|u\|_{p;\Omega} + \|f\|_{p;\Omega}\right),$$

where C depends on $n, p, \lambda, \Lambda, \Omega', \Omega$ and the moduli of continuity of the coefficients a_{ij} on Ω'.

The next theorem is an extension of the Calderon-Zygmund inequality to the half-space. Let

$$\begin{aligned} \Omega^+ &= \Omega \cap \mathbb{R}^n_+ = \{x \in \Omega \,:\, x_n > 0\}, \\ (\partial\Omega)^+ &= (\partial\Omega) \cap \mathbb{R}^n_+ = \{x \in \partial\Omega \,:\, x_n > 0\}. \end{aligned}$$

We formulate the following extension of Theorem 2.2.2.

Theorem 2.2.4. *Let $u \in W^{1,1}_0(\Omega^+)$, $f \in L^p(\Omega)^+$, $1 < p < \infty$, satisfy $\Delta u = f$ weakly in Ω^+ with $u = 0$ near $(\partial\Omega)^+$. Then $u \in W^{2,p}(\Omega^+) \cap W^{1,p}_0(\Omega)$ and*

$$\|D^2u\|_{p;\Omega^+} \leq C\|f\|_{p;\Omega^+} \quad \text{where} \quad C = C(n,p).$$

For the global estimate we require that the boundary values are taken in the sense of $W^{1,p}(\Omega)$. If T is a subset of $\partial\Omega$ and $u \in W^{1,p}(\Omega)$, we say that $u = 0$ on T in the sense of $W^{1,p}(\Omega)$ if u is the limit in $W^{1,p}(\Omega)$ of a sequence of functions in $C^1(\Omega)$ vanishing near T. When u is continuous on T it is implied by u vanishing on T in the usual pointwise sense.

Theorem 2.2.5. *Let Ω be a domain in $\mathbb{R}^n$ with a $C^{1,1}$ boundary portion $T \subset \partial\Omega$. Let $1 < p < \infty$ and $u \in W^{2,p}(\Omega)$ be a strong solution of $Lu = f$ in Ω with $u = 0$ on T, in the sense of $W^{1,p}(\Omega)$, where L satisfies (2.2.2) with $a_{ij} \in C(\Omega \cup T)$. Then, for any domain $\Omega' \Subset \Omega \cup T$,*

$$\|u\|_{2,p;\Omega'} \leq C\left(\|u\|_{p;\Omega} + \|f\|_{p;\Omega}\right) \tag{2.2.3}$$

where C depends on $n, p, \lambda, \Omega', \Omega$ and the moduli of continuity of the coefficients a_{ij} on Ω'.

The following regularity result plays an important role.

Theorem 2.2.6. *In addition to the hypothesis of Theorem 2.2.5, suppose that $f \in L^q(\Omega)$ for some $q \in (p, \infty)$. Then, $u \in W^{2,q}_{loc}(\Omega \cup T)$, $u = 0$ on T in the sense of $W^{1,q}(\Omega)$, and consequently, u satisfies the estimate* (2.2.3) *with p replaced by q.*

When $T = \partial\Omega$ we may take $\Omega' = \Omega$ to obtain a global $W^{2,p}(\Omega)$ estimate. In [48] the following refinement is proved.

Theorem 2.2.7. *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$ and suppose the operator L satisfies the conditions* (2.2.2) *with $a_{ij} \in C(\bar{\Omega})$, $i, j = 1, \ldots, n$. Then if $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$, $1 < p < \infty$, we have*

$$\|u\|_{2,p;\Omega} \leq C\|Lu - \sigma u\|_{p;\Omega}$$

for all $\sigma \geq \sigma_0$, where C and σ_0 are positive constants depending only on $n, p, \lambda, \Lambda, \Omega$ and the moduli of continuity of the coefficients a_{ij}.If $c \leq 0$, we can take the constant $\sigma = 0$.

The next theorem concerns the higher-order regularity data.

Theorem 2.2.8. *Let u be a $W^{2,p}_{loc}(\Omega)$ solution of the elliptic equation $Lu = f$ in a domain Ω, where*

$$a_{ij},\ b_i, c\ \in C^{k-1,1}(\Omega)\ \left(C^{k-1,\alpha}(\Omega)\right), \quad f \in W^{k,q}_{loc}(\Omega),\ \left(C^{k-1,\alpha}(\Omega)\right)$$

with $1 < p, q < \infty$, $k \geq 1$, $0 < \alpha < 1$. Then

$$u \in W^{k+2,q}_{loc}(\Omega)\ \left(C^{k+1,\alpha}(\Omega)\right).$$

Furthermore, if L is strictly elliptic in Ω and

$$a_{ij},\ b_i, c\ \in C^{k-1,1}(\bar{\Omega})\ \left(C^{k-1,\alpha}(\bar{\Omega})\right), \quad \Omega \in C^{k+1,1}\ \left(C^{k+1,\alpha}\right),$$

then

$$u \in W^{k+2,q}(\Omega)\ \left(C^{k+1,\alpha}(\bar{\Omega})\right).$$

2.2.2 Pointwise Estimates

The crown jewels of the theory of strong solutions of second-order elliptic equations are various pointwise estimates.

In the following, we assume that the coefficients a_{ij}, b_i, c of the operator L are measurable, L is elliptic in Ω and

$$|b|/\mathcal{D}^*, \quad f/\mathcal{D}^* \in L^n(\Omega), \quad c \leq 0 \quad \text{in} \quad \Omega.$$

The following theorem is the maximum principle of Alexandrov, which is often referred to as Alexandrov-Bakelman-Pucci estimate, or ABP. A good survey on Alexandrov's maximum principle, both in elliptic and parabolic cases, can be found in [94]. For the majority of applications the form given in [48] is sufficient.

Theorem 2.2.9. *Let $Lu = f$ in a bounded domain Ω and $u \in C(\bar{\Omega}) \cap W^{2,n}_{loc}(\Omega)$. Then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C\|f/\mathcal{D}^*\|_{L^n(\Omega)}$$

where C is a constant depending only on n, diamΩ *and* $\|b/\mathcal{D}^*\|_{L^n(\Omega)}$.

If u is not assumed continuous on $\bar{\Omega}$ the conclusion of this theorem can be modified by replacing $\sup_{\partial\Omega} u^+$ by $\limsup_{x\to\partial\Omega} u^+$. The next three theorems are direct consequences of Alexandrov's maximum principle.

Theorem 2.2.10 (Uniqueness). *Suppose that u and v are two functions in $C(\bar{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ satisfying $Lu = Lv$ in Ω and $u = v$ on $\partial\Omega$. Then $u = v$ in Ω.*

The following lemma is one of the classical results of the qualitative theory of elliptic equations. Its proof relies on comparison with specially chosen barrier functions. For the proof and discussions I refer the reader to [48] or [86].

Theorem 2.2.11 (Hopf-Oleinik lemma). *Let $u \in W^{2,n}_{loc}(\Omega)$ satisfy $Lu \leq 0$ in Ω with $c \equiv 0$. Let (i) $x_0 \in \partial\Omega$, (ii) $\partial\Omega$ be of class $C^{1,1}$ at x_0, (iii) u be continuous at x_0 and (iv) $u(x_0) < u(x)$ for any $x \in \Omega \cap B_\varepsilon(x_0)$. Let $l \in \mathbb{R}^n$ be the inner normal to $\partial\Omega$ at x_0. Then $\frac{\partial u}{\partial l}(x_0) > 0$.*

Theorem 2.2.12 (Strong maximum principle). *If $u \in W^{2,n}_{loc}(\Omega)$ satisfies $Lu \geq 0$ in Ω and $c = 0$ ($c \leq 0$), then u cannot achieve a maximum (nonnegative maximum) in Ω unless it is constant.*

Next, assume that the operator is strictly elliptic with bounded coefficients in Ω, and fix constants γ and ν such that

$$\frac{\Lambda}{\lambda} \leq \gamma, \quad \left(\frac{|b|}{\lambda}\right)^2, \ \frac{|c|}{\lambda} \leq \nu.$$

The following theorem is a local pointwise estimate for strong solutions, as opposed to the global estimate given by the Alexandrov's maximum principle. It is surprising that, in fact, this theorem can be obtained as a consequence of the Alexandrov's estimate.

Theorem 2.2.13. *Let $u \in W^{2,p}(\Omega)$ and suppose $Lu \geq f$, where $f \in L^n(\Omega)$. Then for any ball $B = B_{2R}(y) \subset \Omega$ and $p > 0$, we have*

$$\sup_{B_R(y)} u \leq C\left[\left(\frac{1}{|B|}\int_B (u^+)^p dx\right)^{1/p} + \frac{R}{\lambda}\|f\|_{L^n(B)}\right]$$

where $C = C(n, p, \gamma, \nu R^2)$.

There is an analogous infimum estimate for nonnegative supersolutions. Note, that in the infimum estimate, the exponent p depends on L and cannot be taken arbitrary.

Theorem 2.2.14. *Let* $u \in W^{2,n}(\Omega)$ *satisfy* $Lu \le f$ *in* Ω, *where* $f \in L^n(\Omega)$, *and suppose that* u *is non-negative in a ball* $B = B_{2R}(y) \subset \Omega$. *Then*

$$\left(\frac{1}{|B_R|}\int_{B_R} u^p dx\right)^{1/p} \le C\left(\inf_{B_R(y)} u + \frac{R}{\lambda}\|f\|_{L^n(B)}\right)$$

where p *and* C *are positive constants depending only on* n, γ *and* νR^2.

It is easy to see that a combination of two preceding theorems yields a full Harnack inequality.

Theorem 2.2.15. *Let* $u \in W^{2,n}(\Omega)$ *satisfy* $Lu = 0$, $u \ge 0$ *in* Ω. *Then for any ball* $B_{2R}(y) \subset \Omega$, *we have*

$$\sup_{B_R(y)} u \le C \inf_{B_R(y)} u$$

where $C = C(n, \gamma, \nu R^2)$.

Moreover, using Theorem 2.2.13 it is not hard to prove the Hölder estimates for operators in general form.

Theorem 2.2.16. *Let* $u \in W^{2,n}(\Omega)$ *satisfy the equation* $Lu = f$ *in* Ω. *Then, for any ball* $B = B_{R_0}(y) \subset \Omega$ *and* $R \le R_0$, *we have*

$$\operatorname*{osc}_{B_R(y)} u \le C\left(\frac{R}{R_0}\right)^{\alpha}\left(\operatorname*{osc}_{B_0} u + R_0\|f - cu\|_{n;B_0}\right),$$

where $C = C(n, \gamma, \nu R_0^2)$, $\alpha = \alpha(n, \gamma, \nu R_0^2)$ *are positive constants.*

The local maximum principle (2.2.13) may be extended to balls intersecting the boundary as follows.

Theorem 2.2.17. *Let* $u \in W^{2,n}(\Omega) \cap C(\bar{\Omega})$ *satisfy* $Lu \ge f$ *in* Ω, $u \le 0$ *on* $B \cap \partial\Omega$ *where* $f \in L^n(\Omega)$ *and* $B = B_{2R}(y)$ *is a ball in* $\mathbb{R}^n$. *Then, for any* $p > 0$, *we have*

$$\sup_{\Omega \cap B_R(y)} u \le C\left[\left(\frac{1}{|B|}\int_{B\cap\Omega} (u^+)^p dx\right)^{1/p} + \frac{R}{\lambda}\|f\|_{L^n(\Omega)}\right],$$

where $C = C(n, \gamma, \nu R^2, p)$.

The infimum estimate of Theorem 2.2.14 also admits an extension to the boundary.

Theorem 2.2.18. *Let $u \in W^{2,n}(\Omega)$ satisfy $Lu \leq f$ in Ω, $u \geq 0$ in $B \cap \Omega$, where $B = B_{2R}(y)$ is a ball in $\mathbb{R}^n$. Set $m = \inf_{B\cap\partial\Omega} u$ and*

$$u_m^-(x) = \begin{cases} \inf(u(x), m) & \text{for} \quad x \in B \cap \Omega \\ m \quad \text{for} \quad x \in B \setminus \Omega. \end{cases}$$

Then

$$\left(\frac{1}{|B_R|} \int_{B_R} (u_m^-)^p \right)^{1/p} \leq C \left(\inf_{\Omega \cap B_R} u + \frac{R}{\lambda} \|f\|_{L^n(B\cap\Omega)} \right),$$

where p and C are positive constants depending only on n, γ and νR^2. If we assume only $u \in W^{2,n}_{loc}(\Omega)$, then the conclusion of this theorem holds with $m = \liminf_{x \to B\cap\partial\Omega} u$.

Analogously to the interior Hölder estimate of Theorem 2.2.16, we can state now the Hölder estimate up to the boundary.

Theorem 2.2.19. *Let $u \in W^{2,n}_{loc}(\Omega)$ satisfy $Lu = f$ in Ω where $f \in L^n(\Omega)$, and suppose that Ω satisfies an exterior cone condition at a point $y \in \partial\Omega$. Then, for any $0 < R < R_0$ and ball $B_0 = B_{R_0}(y)$, we have*

$$\underset{\Omega\cap B_R}{osc}\, u \leq C \left[\left(\frac{R}{R_0} \right)^\alpha \left(\underset{\Omega\cap B_{R_0}}{osc}\, u + R_0 \|f - cu\|_{n;\Omega\cap B_{R_0}} \right) + \sigma\left(\sqrt{RR_0}\right) \right],$$

where $C = C(n, \gamma, \nu R_0^2, V_y)$, $\alpha = \alpha(n, \gamma, \nu R_0^2, V_y)$ are positive constants, V_y is the exterior cone at y and

$$\sigma(r) = \underset{\partial\Omega\cap B_r}{osc} = \limsup_{x\to\partial\Omega\cap B_r} u - \liminf_{x\to\partial\Omega\cap B_r} u$$

for $0 < r \leq R_0$.

Combined, Theorems 2.2.16 and 2.2.19 give the uniform Hölder estimate up to the boundary.

Theorem 2.2.20. *Let $u \in W^{2,n}(\Omega) \cap C(\bar{\Omega})$ satisfy*

$$Lu = f \quad in \quad \Omega, \qquad u = \phi \quad on \quad \partial\Omega$$

where $f \in L^n(\Omega)$, $\phi \in C^\beta(\bar{\Omega})$ for some $\beta > 0$, and suppose that $\partial\Omega$ satisfies a uniform exterior cone condition. Then $u \in C^\alpha(\bar{\Omega})$ and

$$|u|_{\alpha;\Omega} \leq C$$

where α and C are positive constants depending on $n, \gamma, \nu, \beta, \Omega, |\phi|_{\beta,\Omega}$ and $\sup_\Omega |u|$.

The next (Krylov's) theorem provides an interesting and useful Hölder estimate for the trace of the gradient of a solution on the boundary. Assume now that the operator L is uniformly elliptic and does not contain lower-order terms, $Lu = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j}$.

Theorem 2.2.21. *Let $u \in W^{2,n}_{loc}(B^+) \cap C(\overline{B^+})$ satisfy the equation $Lu = f$ in the half ball $B^+ = B_{R_0}(0) \cap \mathbb{R}^n_+$ with $f \in L^\infty(B^+)$ and $u = 0$ on $T = B_{R_0}(0) \cap \partial\mathbb{R}^n_+$. Then for any $R \le R_0$ we have*

$$\operatorname*{osc}_{B_R^+} \frac{u}{x_n} \le C\left(\frac{R}{R_0}\right)^\alpha \left(\operatorname*{osc}_{B^+} \frac{u}{x_n} + R_0 \sup_{B^+} \frac{|f|}{\lambda}\right)$$

where α and C are positive constants depending only on n and γ.

2.2.3 Existence Results

The central existence result is the following theorem.

Theorem 2.2.22. *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$ and let the operator L be strictly elliptic in Ω with coefficients*

$$a_{ij} \in C(\bar{\Omega}), \quad b_i, \ c \in L^\infty(\Omega) \quad \textit{and} \quad c \le 0.$$

Then, if $f \in L^p(\Omega)$ and $\phi \in W^{2,p}(\Omega)$ with $1 < p < \infty$, the Dirichlet problem

$$Lu = f \quad \textit{in} \quad \Omega, \quad u - \phi \in W^{1,p}_0(\Omega)$$

has a uniques solution $u \in W^{2,p}(\Omega)$.

When $p > n/2$, we obtain an existence theorem for continuous boundary values.

Theorem 2.2.23. *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$, and let the operator L be strictly elliptic in Ω with coefficients*

$$a_{ij} \in C(\bar{\Omega}), \quad b_i, \ c \in L^\infty(\Omega) \quad \textit{and} \quad c \le 0.$$

Then if $f \in L^p(\Omega)$, $p > n/2$, and $\phi \in C(\partial\Omega)$, the Diriclet problem

$$Lu = f \quad \textit{in} \quad \Omega, \quad u = \phi \quad \textit{on} \quad \partial\Omega,$$

has a unique solution $u \in W^{2,p}_{loc}(\Omega) \cap C(\bar{\Omega})$.

With the help of Theorems 2.2.16 and 2.2.19 it is possible to prove the following variant of the existence theorem 2.2.23. The difference is that we can work with 'bad' domains now (for example, Lipshitz), but the price paid for it is the higher integrability of the data.

Theorem 2.2.24. *Let L be strictly elliptic in a bounded domain Ω with coefficients*

$$a_{ij} \in C(\Omega) \cap L^\infty(\Omega), \quad b_i, \; c \in L^\infty(\Omega),$$

and suppose that Ω satisfies an exterior cone condition at every boundary point. Let ϕ be a continuous function on $\partial\Omega$. Then if $f \in L^p(\Omega)$, $p \geq n$, the Dirichlet problem

$$Lu = f \quad in \quad \Omega, \quad u = \phi \quad on \quad \partial\Omega$$

has a unique solution $u \in W^{2,p}_{loc}(\Omega) \cap C(\bar{\Omega})$.

In the scale of the Hölder spaces, the existence theorems are somewhat more complete, but they require the higher regularity of the data and the domain.

Theorem 2.2.25. *Let L be strictly elliptic in a bounded domain $\Omega \subset \mathbb{R}^n$ with $c \leq 0$, and let f and the coefficients of L be bounded and belong to $C^\alpha(\Omega)$. Suppose that Ω satisfies an exterior sphere condition at every boundary point. Then, if φ is continuous on $\partial\Omega$, the Dirichlet problem*

$$Lu = f \quad in \quad \Omega, \quad u = \varphi \quad on \quad \partial\Omega$$

has a unique solution $u \in C(\bar{\Omega}) \cap C^{2,\alpha}(\Omega)$.

Theorem 2.2.26. *Let L be strictly elliptic in a bounded domain $\Omega \subset \mathbb{R}^n$ with $c \leq 0$, and let f and the coefficients of L belong to $C^\alpha(\Omega)$. Suppose that Ω is a $C^{2,\alpha}$ domain and that $\varphi \in C^{2,\alpha}(\bar{\Omega})$. Then the Dirichlet problem*

$$Lu = f \quad in \quad \Omega, \quad u = \varphi \quad on \quad \partial\Omega$$

has a unique solution $u \in C^{2,\alpha}(\bar{\Omega})$.

2.3 Nonlinear Parabolic Equations

In this section I gathered some results from [25].

2.3.1 Common Tools

The main tool in the regularity theory of divergent PDEs is Sobolev spaces and imbedding theorems. In the parabolic case, we need multiplicative imbedding inequalities of the Gagliardo-Nirenberg type.

Theorem 2.3.1. *Let $v \in W_0^{1,p}(\Omega)$, $p \geq 1$, $\Omega \subset \mathbb{R}^n$. For every fixed number $s \geq 1$ there exists a constant C depending only on n, p and s such that*

$$\|v\|_{q,\Omega} \leq C \|Dv\|^\alpha_{p;\Omega} \|v\|^{1-\alpha}_{s;\Omega},$$

where $\alpha \in [0,1]$, $p, q \geq 1$, *are linked by*

$$\alpha = \left(\frac{1}{s} - \frac{1}{q}\right)\left(\frac{1}{n} - \frac{1}{p} - \frac{1}{s}\right)^{-1},$$

and their admissible range is:

1)If $n = 1$,

$$q \in [s, \infty], \quad \alpha \in \left[0, \frac{p}{p + s(p-1)}\right];$$

2) If $1 \leq p < n$, *then* $\alpha \in [0,1]$ *and*

$$q \in \left[s, \frac{np}{n-p}\right] \quad \text{if} \quad s \leq \frac{np}{n-p},$$
$$q \in \left[\frac{np}{n-p}, s\right] \quad \text{if} \quad s \geq \frac{np}{n-p};$$

3) If $p \geq n > 1$, *then*

$$q \in [s, \infty) \quad \text{and} \quad \alpha \in \left[0, \frac{np}{np + s(p-n)}\right).$$

The next inequality is usually referred to as the De Giorgi-Poincaré type inequality.

Theorem 2.3.2. *Let* Ω *be a bounded convex set in* $\mathbb{R}^n$ *and let* $\varphi \in C(\bar{\Omega})$ *satisfy*

$$\begin{cases} 0 \leq \varphi \leq 1 \quad \text{for all} \quad x \in \Omega, \\ \text{the sets} \quad [\varphi > k] \quad \text{are convex} \quad \text{for all} \quad k \in (0,1). \end{cases}$$

Let $v \in W^{1,p}(\Omega)$, $p \geq 1$, *and assume that the set*

$$\mathcal{E} = [v = 0] \cap [\varphi = 1]$$

has positive measure. There exists a constant C *depending only on* n, p *and independent of* v *and* φ *such that*

$$\left(\int_\Omega \varphi |v|^p \, dx\right)^{\frac{1}{p}} \leq C \frac{(diam\, \Omega)^n}{|\mathcal{E}|^{\frac{n-1}{n}}} \left(\int_\Omega |Dv|^p \, dx\right)^{\frac{1}{p}}.$$

If we set in this theorem $\varphi = 1$, $p = 1$ and apply it to the function

$$w = \begin{cases} \min(v, l) - k & \text{if} \quad v > k, \\ 0 & \text{if} \quad v \leq k \end{cases}$$

we obtain the following useful corollary.

Theorem 2.3.3. *Let $\Omega \subset B_\rho(x_0) \subset \mathbb{R}^n$ be a convex domain, $v \in W^{1,1}(\Omega) \cap C(\Omega)$ and let k and l be any pair of real numbers such that $k < l$. There exists a constant γ depending only on n, p and independent of k, l, v, x_0, p such that*

$$(l-k)|[v > l]| \leq \gamma \frac{\rho^{n+1}}{|[v < k]|} \int_{[k<v<l]} |Dv|\, dx.$$

The next lemma is a simple but very powerful tool recently discovered in [27]

Theorem 2.3.4. *Let $K\rho(y) \subset \mathbb{R}^n$ denote a cube of edge ρ centered at y. Let $u \in W^{1,1}(K_\rho(0))$ satisfy*

$$\int_{K_\rho(0)} |Du|\, dx \leq \gamma\rho^{n-1} \quad \textit{and} \quad |[u > 1]| > \alpha |K_\rho(0)|$$

for some $\gamma > 0$ and $\alpha \in (0,1)$. Then for every $\delta \in (0,1)$ and $\lambda \in (0,1)$ there exist $x_0 \in K_\rho(0)$ and $\eta = \eta(\alpha, \delta, \gamma, \lambda, n) \in (0,1)$ such that

$$|[u > \lambda] \cap K_{\eta\rho}(x_0)| > (1-\delta)|K_{\eta\rho}(x_0)|.$$

The next lemma on fast geometric convergence is due to Ladyzhenskaja and Uraltseva.

Lemma 2.3.5. *Let $\{Y_n\}$, $n = 0, 1, 2\ldots$ be a sequence of positive numbers, satisfying the recursive inequality*

$$Y_{n+1} \leq C b^n Y_n^{1+\alpha}$$

where $C, b > 1$ and $\alpha > 0$ are given numbers. If

$$Y_0 \leq C^{-1/\alpha} b^{-1/\alpha^2},$$

then $\{Y_n\}$ converges to zero as $n \to \infty$.

2.3.2 Parabolic spaces and embeddings.

Let Ω be a bounded domain in $\mathbb{R}^n$ and $T > 0$. By Ω_T denote $\Omega \times (0,T)$. Let $q, r \geq 1$. We say that a function f defined and measurable in Ω_T belongs to

$$L^{q,r}(\Omega) = L^r(0,T; L^q(\Omega))$$

if

$$\|f\|_{q,r;\Omega_T} = \left(\int_0^T (|f|^q\, dx)^{\frac{r}{q}}\, d\tau \right)^{\frac{1}{r}} < \infty.$$

Let $m, p \geq 1$ and consider the Banach spaces

$$V^{m,p}(\Omega_T) = L^\infty(0,T; L^m(\Omega)) \cap L^p(0,T; W^{1,p}(\Omega))$$

and

$$V_0^{m,p}(\Omega_T) = L^\infty(0,T;L^m(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega)),$$

both equipped with the norm

$$\|v\|_{V^{m,p}(\Omega_T)} = \underset{0<t<T}{esssup} \|v(\cdot,t)\|_{m,\Omega} + \|Dv\|_{p,\Omega_T}.$$

Both spaces are embedded in $L^q(\Omega_T)$ for some $q > p$.

Theorem 2.3.6. *There exists a constant γ depending only on n, p and m such that for every $v \in V_0^{m,p}(\Omega_T)$*

$$\iint_{\Omega_T} |v(x,t)|^q \, dxdt$$

$$\leq \gamma^q \left(\iint_{\Omega_T} |Dv(x,t)|^p \, dxdt \right) \left(\underset{0<t<T}{esssup} \int_\Omega |v(x,t)|^m \, dx \right)^{\frac{p}{n}},$$

where

$$q = p\frac{n+m}{n}.$$

Moreover,

$$\|v\|_{q;\Omega_T} \leq \gamma \|v\|_{V^{m,p}(\Omega_T)}.$$

Remark. The result of Theorem 2.3.6 continues to hold for functions $v \in V^{m,p}(\Omega_T)$ such that

$$\int_\Omega v(x,t) \, dx = 0 \quad \text{for a.e.} \quad t \in (0,T),$$

provided that $\partial\Omega$ is sufficiently smooth.

For the embeddings into the parabolic spaces $L^{q,r}(\Omega_T)$ there is a more precise result.

Theorem 2.3.7. *There exists a constant γ depending only on n and p such that for every $v \in V_0^{p,p}(\Omega_T)$,*

$$\|v\|_{q,r;\Omega_T} \leq \gamma \|v\|_{V^{p,p}(\Omega_T)},$$

where the numbers $q, r \geq 1$ are linked by

$$\frac{1}{r} + \frac{n}{pq} = \frac{n}{p^2},$$

and their admissible range is

$$\begin{cases} q \in (p,\infty], & r \in [p^2,\infty) \quad \text{if} \quad n = 1, \\ q \in \left[p, \frac{np}{n-p}\right], & r \in [p,\infty] \quad \text{if} \quad 1 \leq p < n, \\ q \in [p,\infty], & r \in (\frac{p^2}{n},\infty] \quad \text{if} \quad 1 < n \leq p. \end{cases}$$

To obtain energy inequalities we need to work with truncations and Steklov averages of functions from the Sobolev and Lebesgue spaces.

Lemma 2.3.8. *Let $v \in V^{m,p}(\Omega_T)$. Then $(v-k)_\pm \in V^{m,p}(\Omega_T)$ for all $k \in \mathbb{R}$. Assume in addition that the trace of $x \to v(x,t)$ on $\partial\Omega$ is essentially bounded and*

$$\operatorname*{ess\,sup}_{0<t<T} \|v(\cdot,t)\|_{\infty,\partial\Omega} \le k_0, \quad \text{for some} \quad k_0 > 0.$$

Then $(v-k)_\pm \in V_0^{m,p}(\Omega_T)$ for all $k \ge k_0$.

Lemma 2.3.9. *Let $v_i \in L^p(0,T;W^{1,p}(\Omega))$ for $i = 1,\dots,i_0 \in \mathbb{N}$. Then*

$$w = \min(v_1, v_2, \dots, v_n) \in L^p(0,T;W^{1,p}(\Omega)).$$

Let v be a function in $L^1(\Omega_T)$ and for $0 < h < T$ introduce the Steklov averages $v_h(\cdot,t)$ defined for all $0 < t < T$ by

$$v_h = \begin{cases} \frac{1}{h}\int_t^{t+h} v(\cdot,\tau)\,d\tau, & t \in (0,T-h], \\ 0, & t > T-h; \end{cases}$$

$$v_{\bar{h}} = \begin{cases} \frac{1}{h}\int_{t-h}^{t} v(\cdot,\tau)\,d\tau, & t \in (h,T], \\ 0, & t < h; \end{cases}$$

Lemma 2.3.10. *Let $v \in L^{q,r}(\Omega_T)$. Then*

$$v_h \to v \quad in \quad L^{q,r}(\Omega_{T-\varepsilon})$$

as $h \to 0$ for every $\varepsilon \in (0,T)$. If $v \in C(0,T;L^q(\Omega))$, then

$$v_h(\cdot,t) \to v(\cdot,t) \quad in \quad L^q(\Omega)$$

as $h \to 0$ for every $\varepsilon \in (0,T)$ and $t \in (0,T-\varepsilon)$. A similar statement holds for $v_{\bar{h}}$.

Chapter 3

Semilinear Elliptic Equations

3.1 Estimates for Emden-Fowler type inequalities with absorption term.

This section follows line by line my paper [105].

1. Introduction, main results and applications

In this paper we study the behaviour of solutions to the Emden-Fowler type inequality with absorbtion term:

$$sign(u)\mathcal{L}u \geq \frac{c(x)}{|x|^2}|u|^\sigma, \quad x \in \Omega \subset \mathbb{R}^n, \tag{3.1.1}$$

where Ω is either the exterior of a ball or a ball punctured at the origin. In (3.1.1) $\sigma > 1$ is a constant and $\mathcal{L}$ is the elliptic operator in non-divergence form

$$\mathcal{L}u = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{n} b_i(x)u_{x_i}. \tag{3.1.2}$$

The coefficients $a_{ij}, b_i, c : \Omega \to \mathbb{R}$ are locally bounded measurable functions satisfying the conditions

(i) there exist $\nu_1, \nu_2 > 0$ such that

$$\nu_1|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \nu_2|\xi|^2 \text{ for all } x \in \Omega,\ \xi \in \mathbb{R}^n, \tag{3.1.3}$$

(ii) $\sup\limits_{x\in\Omega}\left|\sum\limits_{i=1}^{n} b_i(x)x_i\right| < +\infty$, and $\quad$ (3.1.4)

(iii) $\inf\limits_{x\in K} c(x) > 0$ for any compact subset $K \Subset \Omega$. $\quad$ (3.1.5)

We say that $u \in W^{2,n}_{loc}(\Omega)$ is a solution to (3.1.1) in Ω if u satisfies equation (3.1.1) pointwise almost everywhere (a.e.) in Ω. Similarly, $u \in W^{2,n}_{loc}(\Omega)$ is a supersolution to (3.1.1) in Ω if u satisfies

$$sign(u)\mathcal{L}u \le \frac{c(x)}{|x|^2}|u|^\sigma \text{ a.e. in } \Omega.$$

By $B_r(x)$ we denote the ball of radius r with the centre at point x, i.e. $B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$, $B_r := B_r(0)$. A_{ρ_1,ρ_2} stands for an open annulus $A_{\rho_1,\rho_2} = \{x \in \mathbb{R}^n : \rho_1 < |x| < \rho_2\}$, where $\rho_1, \rho_2 \in [0, +\infty]$. Thus, $A_{\rho,\infty} = \mathbb{R}^n/\overline{B_\rho}$ and $A_{0,\rho} = B_\rho/\{0\}$.

We introduce the following functions which are used to state the properties of $\mathcal{L}$:

$$T(x) = \sum_{i=1}^n a_{ii}(x) + \sum_{i=1}^n b_i(x)x_i, \quad \Phi(x) = \sum_{i,j=1}^n a_{ij}(x)\frac{x_i x_j}{|x|^2}, \quad A(x) = \frac{T(x)}{\Phi(x)}.$$

The last function was introduced in [85] where it was called the "effective dimension". It turns out that it plays an important role not only in describing properties of the corresponding linear equation but also in studying nonlinear equations of Emden-Fowler type ([67], [68]). The following notation is standard

$$u_+ = \max(u,0), \quad u_- = \max(-u,0).$$

We also denote $M(r) = \sup_{|x|=r} u(x)$.

Inequalities of type (3.1.1) are of great importance in many areas of mathematical physics and for a long time have been attracting attention of many authors. The qualitative theory of this type of equations has a rich mathematical structure and yields a lot of beautiful results. One of the interesting and popular questions in this theory is a study of singularities of solutions to equations and inequalities of type (3.1.1) and their behavior in exterior domains. The tool whose value is hard to overestimate is widely known as the Keller-Osserman estimate. For the equation of the form

$$\Delta u = u^\sigma \text{ in } \Omega \tag{3.1.6}$$

it was first established in the works of the named authors [57, 98] and reads as follows. Suppose $u \in C^2(\Omega)$ is a solution to (3.1.6). Then there exists a constant $C = C(\sigma, n)$ such that

$$|u(x)| \le C\left(dist(x,\partial\Omega)\right)^{\frac{2}{1-\sigma}}. \tag{3.1.7}$$

If u is a solution to (3.1.6) in $A_{\rho,\infty}$, the last inequality immediately implies that $u(x) \to 0$ as $x \to \infty$, and

$$|u(x)| \le C|x|^{\frac{2}{1-\sigma}}, \quad x \in A_{2\rho,\infty}.$$

Property (3.1.7) of solutions of (3.1.6) is a feature inherent to this class of nonlinear equations and with its help many results concerning behaviour of solutions of (3.1.6) are derived and obtaining existence results is significantly simplified. Moreover, in some cases this *a priori* bound leads to the removability of isolated singularities ([8],[61]) or to their fairly complete description (see [69, 113, 114, 117] and references therein). This estimate was generalized to divergent and non-divergent elliptic operators of the form $\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$, $a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}$ respectively in [61] and for parabolic equations and inequalities in [21]. The latter work studies even more general case of the nonlinearity $f(u)$ with the function f satisfying certain structure conditions. In both [61] and [21] the differential operators do not contain any lower order terms or weight standing by nonlinearity. To some extent, via a use of the scaling method, the weight in front of the nonlinearity can be tackled. We show now the point at which the problems arise.

For simplicity, let u be a positive solution to the equation

$$\Delta u = \frac{c(x)}{|x|^2} u^\sigma \tag{3.1.8}$$

in $A_{\rho,\infty}$. For $x \in A_{\delta\rho,\infty}, \delta > 1$, consider the ball $B := B_{|x|/\delta}(x)$. Substituting $u = \beta v$ we see that in this ball

$$\Delta v \geq \beta^{\sigma-1}\left(\inf_{y\in B}\frac{c(y)}{|y|^2}\right) v^\sigma = v^\sigma,$$

if we choose $\beta = \left(\inf_{y\in B}\frac{c(y)}{|y|^2}\right)^{\frac{1}{1-\sigma}}$. Now estimate (3.1.7) yields $v(x) \leq C|x|^{\frac{2}{1-\sigma}}$, and hence

$$|u(x)| \leq C\left(\inf_{|y-x|<|x|/\delta} c(y)\right)^{\frac{1}{1-\sigma}}. \tag{3.1.9}$$

We will refer further to this argument as the "scaling argument". If we take $c(x) \equiv 1$, in this way we obtain only boundedness and if $c(x) \to 0$ as $x \to \infty$ estimate (3.1.9) may fail to produce the right answer as is shown in our examples below. The cases when it happens present often the special interest as "critical cases".

Great work in this direction, and in much greater generality than is present here, was done by A. Kon'kov (see monograph [71], also [70, 73, 74]). His proofs rely on subtle integral estimates and comparison theorems which reduce studying *positive* solutions to (3.1.1) to studying positive solutions to the corresponding ODE of the same type. This approach was carried over to the study of nonlinear parabolic equations and inequalities.

The aim of this paper is to cover the remaining gaps and to obtain the sharp estimates in cases which cannot be covered by the scaling argument. Moreover, we present here a proof which is elementary in nature and relies entirely on the maximum principle ([48, 99]) and explicit construction of supersolutions. The idea in [61] (and in [21]) was to write a supersolution in the form $\sum_{i=1}^{n} v(x_i)$, where v was defined as a solution to the appropriate differential equation. In case when the differential operator contains no lower order terms the function v is obtained as a solution to the differential equation

$$v'' = \lambda v^{\sigma},$$

which can be easily integrated. Of course, when the lower order terms and the weight in front of the nonlinearity are present it becomes more complex and needs a great amount of subtle analysis to deal with. Here we show how to avoid this difficulties by taking only the leading terms of the asymptotic expansion of the corresponding differential equation ([7, 60, 72, 107]). It is very likely that the same method can be extended to parabolic equations and to equations with the nonlinearity in the principal part ([74, 110], etc.) - the first point of interest in this case is obtaining the *a priori* bound for solutions of $u_t = \Delta u - |x|^{-2}u^{\sigma}$. The author plans to continue research in this direction.

Now we are ready to formulate the main results of the paper. In the statements of the theorems of this paper C stands for a constant independent of u, whose value varies from line to line.

Theorem 3.1.1. *Let u be a solution to inequality* (3.1.1) *in $A_{\rho,\infty}$. Let $Q(\cdot) : (\rho, +\infty) \to \mathbb{R}^+$ be such that Q is differentiable and*

$$Q(r) \leq \inf_{|x|=r} c(x),$$

$$\sup_{r>\rho} \left| \frac{Q'r}{Q} \right| < +\infty.$$

a) *If $\int_{\rho}^{+\infty} Q(r)\frac{dr}{r} = +\infty$ then*

$$|u(x)| \leq C \left(\int_{\rho}^{|x|} Q(r)\frac{dr}{r} \right)^{\frac{1}{1-\sigma}}, \quad x \in A_{2\rho,\infty}, \tag{3.1.10}$$

and as a consequence

$$u(x) \to 0 \text{ as } x \to \infty.$$

b) *If $\int_{\rho}^{+\infty} Q(r)\frac{dr}{r} < +\infty$ and Q is bounded then*

$$|u(x)| \leq C \left(\int_{|x|}^{+\infty} Q(r)\frac{dr}{r} \right)^{\frac{1}{1-\sigma}}, \quad x \in A_{2\rho,\infty}. \tag{3.1.11}$$

The next theorem is a generalization of the previous one to the case of non-smooth weight.

Theorem 3.1.2. *Let u be a solution to inequality* (3.1.1) *in $A_{\rho,\infty}$. Let $q(\cdot) : (\rho, +\infty) \to \mathbb{R}^+$ be such that*

$$q(r) \le \inf_{x \in \bar{A}_{r/\delta, r\delta}} c(x) \text{ for some } \delta > 1.$$

a. *If $\int_\rho^{+\infty} q(r) \frac{dr}{r} = +\infty$ and*

$$\sup_{r>\rho} \frac{q(r\delta)}{q(r)} < +\infty$$

then

$$|u(x)| \le C \left(\int_\rho^{|x|} q(r) \frac{dr}{r} \right)^{\frac{1}{1-\sigma}}, \quad x \in A_{2\rho,\infty}. \tag{3.1.12}$$

b. *If $\int_\rho^{+\infty} q(r) \frac{dr}{r} < +\infty$, q is bounded and*

$$\sup_{r>\rho} \frac{q(r)}{q(r\delta)} < +\infty$$

then

$$|u(x)| \le C \left(\int_{|x|}^{+\infty} q(r) \frac{dr}{r} \right)^{\frac{1}{1-\sigma}}, \quad x \in A_{2\rho,\infty}. \tag{3.1.13}$$

The third theorem exposes the effect emerging when the weight $c(x)$ is rapidly growing or decaying. (roughly speaking, faster than any power of $|x|$).

Theorem 3.1.3. *Let u be a solution to inequality* (3.1.1) *in $A_{\rho,\infty}$. Let $Q(\cdot) : (\rho, +\infty) \to \mathbb{R}^+$ be such that Q is twice differentiable and*

$$Q(r) \le \inf_{|x|=r} c(x).$$

a. *Suppose that*

$$\frac{rQ'}{Q} \to +\infty \text{ as } r \to +\infty, \text{ and}$$

$$Q', Q'' \ge 0 \ , \ \sup_{r>2\rho} \frac{QQ''}{(Q')^2} < +\infty.$$

Then

$$u(x) \to 0 \text{ as } x \to \infty$$

and

$$|u(x)| \le C \left(\int_\rho^{|x|} \frac{Q^2}{rQ'} \frac{dr}{r} \right)^{\frac{1}{1-\sigma}}, \quad x \in A_{2\rho,\infty}. \tag{3.1.14}$$

b. *Suppose that*

$$\frac{rQ'}{Q} \to -\infty \text{ as } r \to +\infty, \text{ and}$$

$$Q' < 0,\ Q'' \geq 0\ ,\ \sup_{r>2\rho} \frac{QQ''}{(Q')^2} < +\infty.$$

Then

$$|u(x)| \leq C \left(\int_{|x|}^{\infty} \frac{-Q^2}{rQ'} \frac{dr}{r} \right)^{\frac{1}{1-\sigma}}, \quad x \in A_{2\rho,\infty}. \tag{3.1.15}$$

The last theorem deals with the special case of planar domains.

Theorem 3.1.4. *Let u be a solution to inequality (3.1.1) in $A_{\rho,\infty}$, $\rho > 1$. Let $Q(\cdot) : (\rho, +\infty) \to \mathbb{R}^+$ be such that*

$$Q(r) \leq \inf_{|x|=r} c(x) \text{ and}$$

$$\sup_{r>\rho} \left| \frac{Q' r \ln r}{Q} \right| < +\infty.$$

Let

$$A(x) \equiv 2.$$

a. *Suppose that*

$$\int_{\rho}^{+\infty} Q \ln r \frac{dr}{r} = +\infty.$$

Then

$$|u(x)| \leq C \left(\int_{\rho}^{|x|} Q \ln r \frac{dr}{r} \right)^{\frac{1}{1-\sigma}}, \quad |x| > 2\rho. \tag{3.1.16}$$

b. *Suppose that*

$$\int_{\rho}^{+\infty} Q \ln r \frac{dr}{r} < +\infty.$$

Then

$$|u(x)| \leq C \left(\int_{|x|}^{+\infty} Q \ln r \frac{dr}{r} \right)^{\frac{1}{1-\sigma}}, \quad |x| > 2\rho. \tag{3.1.17}$$

Remark 3.1.5. As the results for the exterior of the ball and for the punctured ball are essentially the same, we give only the proof for the former case. Let us note that the assumption of $c(x)$ being locally *strictly* positive is not necessary. We could easily deal with the case of $c(x)$ being only nonnegative and the same results as given here hold. We do not pursue this issue only to avoid unnecessary complication in the notation. For the same reason we confine ourselves to the case of the exterior of the ball - we can easily replace $A_{\rho,\infty}$ in the theorems and their proofs by a domain $\Omega \subset A_{\rho,\infty}$ imposing additional condition $u \mid_{\partial\Omega\cap A_{\rho,\infty}} = 0$. Moreover, it can be easily seen from the method of our proof that we may omit the requirement of *uniform* ellipticity and demand that $\{a_{ij}(x)\}$ be only *locally* uniformly elliptic and its elements be uniformly bounded, thus allowing for degenerate at infinity (or 0) operators.

Theorems 3.1.1 and 3.1.2 are similar and the general idea is contained in Theorem 3.1.1. Theorem 3.1.2 shows how to deal with non-smooth functions through the means of an appropriate regularization. Further, we note that the results of Theorems 3.1.1– 3.1.4 in case **a** imply the uniqueness of solutions of the first boundary value problem for (3.1.1) in $A_{\rho,\infty}$.

Now we provide some examples which illustrate the power of our result.

Example 3.1.6. Natural examples of functions Q which satisfy the conditions of Theorem 3.1.1a can be given by

$$Q_{k,\epsilon}(r) = \left(\Pi_{j=1}^{k-1} \underbrace{\ln \ldots \ln}_{j} r\right)^{-1} \cdot \left(\underbrace{\ln \ldots \ln}_{k} r\right)^{-\varepsilon}, \quad k \geq 1,\ 1 > \varepsilon \geq 0.$$

One can easily check that for $Q_{k,\epsilon}$ estimate (3.1.10) reads as follows

$$|u(x)| \leq C \left(\underbrace{\ln \ldots \ln}_{k} r\right)^{\frac{1-\varepsilon}{1-\sigma}}.$$

Example 3.1.7. If, on the other hand, we choose

$$Q(r) = (\ln r)^{\alpha}, \ \alpha > 0,$$

estimate (3.1.10) yields

$$|u(x)| \leq C\,(\ln r)^{\frac{1+\alpha}{1-\sigma}},$$

which is better than (3.1.9).

Example 3.1.8. Next, if we choose Q such that the conditions of Theorem 3.1.1b are satisfied, a rather natural example is given by

$$Q_{\epsilon}(r) = r^{-\varepsilon}, \ \varepsilon > 0,$$

then it is easy to see that the equation

$$\Delta u = \frac{1}{|x|^{2+\varepsilon}} u^{\sigma}$$

admits a solution growing at infinity

$$u_{\epsilon}(x) = C_{\varepsilon}|x|^{\frac{\varepsilon}{\sigma-1}}.$$

Note that it has the same order of growth as predicted by our estimate.

Remark 3.1.9. If Q has the form $r^\epsilon f(r)$, $\epsilon \neq 0, f'(r) = o(\frac{f}{r})$ as $r \to \infty$ one can easily verify that the estimates (3.1.10), (3.1.11) coincide with (3.1.9). So, the estimate provided by Theorem 3.1.1 presents the main interest when Q has the form

$$\Pi_{j=1}^{k} \left(\underbrace{\ln \ldots \ln}_{j} r \right)^{\alpha_j} .$$

Now we provide an example which demonstrates the sharpness of estimate (3.1.10).

Example 3.1.10. Consider the function $u = (\ln r)^{\frac{1}{1-\sigma}}$ which solves the equation

$$\Delta u - \frac{1}{2} \sum_{j=1}^{2} \frac{x_j u_{x_j}}{|x|^2} = |x|^{-2} \left(\frac{1}{2(\sigma - 1)} + \frac{\sigma}{(\sigma - 1)^2} \frac{1}{\ln r} \right) u^\sigma$$

in $A_{2,\infty} \subset \mathbb{R}^2$. One can easily check that estimate (3.1.10) provides the right answer.

On the other hand, if $c(x) \to \infty$ fast enough (faster then any power of $|x|$) then the scaling argument, as well as estimates (3.1.10), (3.1.11) fail to produce the right answer.

Example 3.1.11. Indeed, set $u(x) = e^{-|x|}$. Then u solves the equation

$$\Delta u - (n-1) \sum_{i=1}^{n} \frac{x_i u_{x_i}}{|x|^2} = \frac{e^{(\sigma-1)|x|} |x|^2}{|x|^2} u^\sigma$$

in $A_{1,\infty}$. The scaling argument provides the estimate

$$|u(x)| \le C e^{-|x|/\delta}, \ \delta > 1.$$

The estimate (3.1.10) yields

$$|u(x)| \le C |x|^{\frac{1}{1-\sigma}} e^{-|x|}$$

which is obviously false, and one can easily check that estimate (3.1.14) gives the right answer

$$|u(x)| \le C e^{-|x|}.$$

The same effect takes place when $c(x) \to 0$ very fast (faster than any power of $|x|$), as is demonstrated by the following example.

Example 3.1.12. Set $u(x) = e^{|x|}$ in $A_{1,\infty}$, which solves the equation

$$\Delta u - (n-1) \sum_{i=1}^{n} \frac{x_i u_{x_i}}{|x|^2} = \frac{e^{(1-\sigma)|x|} |x|^2}{|x|^2} u^\sigma.$$

The estimate provided by the scaling argument is

$$|u(x)| \le C e^{|x|\delta}, \ \delta > 1,$$

whereas estimate (3.1.11) yields

$$|u(x)| \leq C|x|^{\frac{1}{1-\sigma}} e^{|x|},$$

which is clearly not true. Like in the previous example, one can easily check that estimate (3.1.15) provides the correct result

$$|u(x)| \leq C e^{|x|}.$$

Theorem 3.1.4 describes the special case of planar domains. This situation arises when we study the equation of the form

$$\Delta u = \frac{c(x)}{|x|^2} u^\sigma, x \in A_{\rho,\infty} \subset \mathbb{R}^2,$$

and $c(x)$ behaves like a product of logarithmic functions depending on radius. In the three examples below one can easily check that the estimates given by Theorem 3.1.4 yield the right answer.

Example 3.1.13. Let us consider the function $u = (\ln \ln r)^{\frac{1}{1-\sigma}}$ in $A_{10,\infty} \subset \mathbb{R}^2$, which solves the equation

$$\Delta u = \left(\frac{1}{\sigma - 1} (\ln r)^{-2} + \frac{\sigma}{(1-\sigma)^2} (\ln r)^{-2} (\ln \ln r)^{-1} \right) \frac{u^\sigma}{r^2}.$$

Estimate (3.1.10) gives only

$$|u(x)| \leq C\, (\ln r)^{\frac{1}{\sigma - 1}}, \quad |x| > 20,$$

which allows for solutions growing at infinity.

Example 3.1.14. This time we take the function $u = (\ln r)^{\frac{1}{1-\sigma}}$, which solves the equation

$$\Delta u = \frac{\sigma}{(1-\sigma)^2} \frac{1}{r^2 \ln r} u^\sigma \quad \text{in} \quad A_{2,\infty} \in \mathbb{R}^2.$$

If we apply Theorem 3.1.1 we will be left with the estimate

$$|u(x)| \leq C\, (\ln \ln r)^{\frac{1}{1-\sigma}}, \quad |x| > 4,$$

which is again far from optimal.

Example 3.1.15. Let us choose a number k such that $k + 1 > \sigma$. Then the function $u = (\ln r)^{\frac{k}{\sigma - 1}}$ solves the equation

$$\Delta u = \frac{k}{\sigma - 1} \frac{k + 1 - \sigma}{\sigma} (\ln r)^{\frac{-k-2}{\sigma - 1}} \frac{u^\sigma}{r^2}$$

in $A_{2,+\infty} \subset \mathbb{R}^2$. The estimate provided by Theorem 3.1.1 reads as

$$|u(x)| \leq (\ln |x|)^{\frac{k+1}{\sigma - 1}}, \quad |x| \geq 4,$$

and we again observe that in this case we roughly speaking lose one logarithm.

2. Auxiliary facts and central lemma

Some more notation. For a constant $\gamma > 1$ we introduce functions $c_\gamma(\cdot) : (\rho/\gamma, +\infty) \to \mathbb{R}^+$ defined by

$$c_\gamma(r) = \inf_{x \in \bar{A}_{r/\gamma, r\gamma}} c(x).$$

For a constant $\gamma > 1$ and a function $f : (\rho/\gamma, +\infty) \to \mathbb{R}^+$ we define a function $S_\gamma[f] : (\rho, +\infty) \to \mathbb{R}^+$ by

$$S_\gamma[f](r) = \int_{r/\gamma}^{r\gamma} f(s) \frac{ds}{s}.$$

It is clear that if for any $r > \rho/\gamma \quad 0 \le f(r) \le c_\gamma(r)$ then $S_\gamma[f](r) \le 2(\ln\gamma) \inf_{|x|=r} c(x)$ and $|\frac{d}{dr} S_\gamma[f](r)| \le \gamma \frac{\inf_{|x|=r} c(x)}{r}$.

We also note that we often understand a function defined on a subset of positive real semiaxis as a function defined on a subset of $\mathbb{R}^n$. In this case we naturally define $f(x) = f(|x|)$.

Since for technical purposes we will need $q(r)$ to be defined for $r > \rho/\delta$ we set $q(r) = \min(\inf_{\rho \le |x| \le r\delta} c(x))$ for $r \in (\rho/\delta, \rho)$.

When we speak about inf or sup of a function on some set X we mean the intersection of X with the domain of definition of this function.

Maximum principle. In this paper we use a maximum principle in the following form:

Proposition 3.1.16 (Maximum Principle). *Let Ω be a bounded domain and u, v are a solution and a supersolution to* (3.1.1) *in Ω, respectively. If*

$$u_+ \mid_{\partial\Omega} \le v_+ \mid_{\partial\Omega} \quad (u_- \mid_{\partial\Omega} \le v_- \mid_{\partial\Omega}),$$

then

$$u_+ \le v_+ \quad (u_- \le v_-) \text{ in } \Omega.$$

Remark 3.1.17. If two functions $\phi, \psi \in C(\bar{\Omega})$ then there is no problem in understanding the relation $\phi \mid_{\partial\Omega} \le \psi \mid_{\partial\Omega}$. If they are only $C(\Omega)$ then we understand it in the following standard sense:

$$\limsup_{x \to \partial\Omega} (\phi(x) - \psi(x)) \le 0.$$

Corollary 3.1.18. *Let u and v be a solution and a supersolution to* (3.1.1) *in $A_{\rho,\infty}$ and*

$$u, v \to 0 \text{ as } x \to \infty.$$

Then the maximum principle holds in $A_{\rho,\infty}$: If

$$u_+ \mid_{\partial A_{\rho,\infty}} \leq v_+ \mid_{\partial A_{\rho,\infty}} \quad \left(u_- \mid_{\partial A_{\rho,\infty}} \leq v_- \mid_{\partial A_{\rho,\infty}}\right),$$

then

$$u_+ \leq v_+ \quad (u_- \leq v_-) \text{ in } A_{\rho,\infty}.$$

Proof. We first apply the maximum principle to the annulus A_{ρ,ρ_1}, $\rho < \rho_1 < +\infty$ from which we have

$$u_+(x) \leq v_+(x) + \max(u_+, v_+) \mid_{|x|=\rho_1}$$

in A_{ρ,ρ_1}. Now since the last term vanishes as $\rho_1 \to +\infty$, we obtain

$$u_+ \leq v_+ \quad \text{in } A_{\rho,\infty}.$$

$\square$

In the following lemma we may assume in the proof that $\sup_{x\in\partial A_{\rho_1,\rho_2}} |u(x)| < +\infty$. If it is not the case then we first apply this lemmas to a slightly smaller annulus $A_{\rho_1+\varepsilon,\rho_2-\varepsilon}$ and then pass to the limit as $\varepsilon \to 0$.

Lemma 3.1.19. *Let u be a solution to inequality* (3.1.1) *in A_{ρ_1,ρ_2}. Let the functions $R_1(r), R_2(r)$ be given by:*

$$R_1(r) = \int_{\rho_1}^{r} V(s)\frac{ds}{s}, \quad R_2(r) = \int_{r}^{\rho_2} V(s)\frac{ds}{s},$$

where $V(s) \geq 0$; $s \in (\rho_1, \rho_2)$ and $V' \in L_{\infty,loc}(\rho_1, \rho_2)$.

Then in A_{ρ_1,ρ_2} the following estimate holds

$$\sup_{|x|=r} |u(x)| \leq \frac{\sigma+1}{\sigma-1} \max\left(C_1(r) R_1^{\frac{2}{1-\sigma}}(r), C_2(r) R_2^{\frac{2}{1-\sigma}}(r)\right), \tag{3.1.18}$$

where

$$C_1(r) = \left[\sup_{x\in A_{\rho_1,r}} \frac{2\Phi}{\sigma-1} \cdot \left(\frac{\sigma+1}{\sigma-1}\frac{V^2}{c(x)} + R_1\left((2-A(x))\frac{V}{c(x)} - \frac{rV'}{c(x)}\right)_+\right)\right]^{\frac{1}{\sigma-1}}$$

$$C_2(r) = \left[\sup_{x\in A_{r,\rho_2}} \frac{2\Phi}{\sigma-1} \cdot \left(\frac{\sigma+1}{\sigma-1}\frac{V^2}{c(x)} + R_2\left((A(x)-2)\frac{V}{c(x)} + \frac{rV'}{c(x)}\right)_+\right)\right]^{\frac{1}{\sigma-1}}.$$

Proof. First let us recall that for a radial function $f(r)$ the expression for $\mathcal{L}f(r)$ takes the following form

$$\mathcal{L}f(r) = \Phi f'' + \frac{T-\Phi}{r} f'. \tag{3.1.19}$$

Thus, if we want a positive function $f(r)$ to be a supersolution to inequality (3.1.1) it must satisfy the inequality

$$\Phi f'' + \frac{T-\Phi}{r} f' \le \frac{c(x)}{r^2} f^\sigma \quad \text{in} \quad A_{\rho_1,\rho_2}. \tag{3.1.20}$$

Let us introduce the function

$$f(x,y) = x^{\frac{2}{1-\sigma}} - \frac{2}{1-\sigma} y^{\frac{2}{1-\sigma}-1}(x-y) + \frac{2}{\sigma-1} y^{\frac{2}{1-\sigma}}.$$

It is clear that

$$\begin{gathered} f'_x(x,x) = 0, \quad f(x,x) = \frac{\sigma+1}{\sigma-1} x^{\frac{2}{1-\sigma}}, \\ f(x,y) \to +\infty \quad \text{as} \quad x \to 0+, \\ f(x,y) \ge x^{\frac{2}{1-\sigma}}, \quad 0 \le x \le y, \\ f'_x(x,y) = \frac{2}{1-\sigma}\left(x^{\frac{2}{1-\sigma}-1} - y^{\frac{2}{1-\sigma}-1}\right), \\ f''_{xx}(x,y) = \frac{2}{1-\sigma}\left(\frac{2}{1-\sigma} - 1\right) x^{\frac{2\sigma}{1-\sigma}}. \end{gathered}$$

Let us fix some number $\xi \in (\rho_1, \rho_2)$ and denote

$$g_{1,\xi}(r) = f(R_1(r), R_1(\xi)), \quad g_{2,\xi}(r) = f(R_2(r), R_2(\xi)).$$

Let us show that for $C \ge C_1(\xi)$ the function $Cg_{1,\xi}$ is a supersolution to inequality (3.1.1) in $A_{\rho_1,\xi}$. Indeed, for such values of C,

$$\begin{aligned} &\mathcal{L}Cg_{1,\xi} \\ &= C\frac{R_1^{\frac{2\sigma}{1-\sigma}}}{r^2}\frac{2}{\sigma-1}\left[\frac{\sigma+1}{\sigma-1}\Phi V^2 + R_1\left(1 - \left(\frac{R_1(\xi)}{R_1}\right)^{\frac{2}{1-\sigma}-1}\right)((2\Phi - T)V - \Phi V' r)\right] \\ &\le \frac{c(x)}{r^2} g_{1,\xi}^\sigma \cdot C_1^{\sigma-1} C \le \frac{c(x)}{r^2}\left(Cg_{1,\xi}\right)^\sigma. \end{aligned}$$

Analogously, the function $Cg_{2,\xi}$ is a supersolution to inequality (3.1.1) in A_{ξ,ρ_2} for the values $C \ge C_2(\xi)$. Let us introduce now the function

$$F_\xi(r) = \begin{cases} k_1 g_{1,\xi}, & r \le \xi, \\ k_2 g_{2,\xi}, & r \ge \xi, \end{cases}$$

where the constants k_1, k_2 satisfy the conditions

$$\begin{gathered} k_1 g_{1,\xi}(\xi) = k_2 g_{2,\xi}(\xi), \\ k_1 \ge C_1(\xi), \quad k_2 \ge C_2(\xi). \end{gathered}$$

It is obvious that these conditions are satisfied if we choose

$$k_1 = \alpha R_1^{\frac{2}{\sigma-1}}(\xi), \quad k_2 = \alpha R_2^{\frac{2}{\sigma-1}}(\xi),$$
$$\alpha \geq \max\left(C_1(\xi) R_1^{\frac{2}{1-\sigma}}(\xi), C_2(\xi) R_2^{\frac{2}{1-\sigma}}(\xi)\right).$$

Now, $F_\xi(\xi) = \frac{\sigma+1}{\sigma-1}\alpha$, and F is a supersolution to (3.1.1) in A_{ρ_1,ρ_2}. From our construction it is clear that

$$F(r) \to +\infty \text{ as } r \to \rho_1 + 0, \ r \to \rho_2 - 0.$$

We can therefore apply the maximum principle to u and F and immediately obtain the assertion of this lemma. □

In order to move further we need to prove several auxiliary statements.

The proof of the next proposition is very simple and is a direct application of Fubini's theorem, and so we omit it.

Proposition 3.1.20. *Let $f : (\rho/\gamma, +\infty) \to \mathbb{R}^+$ for some $\gamma > 1$. Then $\int^{+\infty} S_\gamma[f](r)\frac{dr}{r}$ and $\int^{+\infty} f(r)\frac{dr}{r}$ converge or diverge simultaneously and*

$$\int_\rho^r S_\gamma[f](s)\frac{ds}{s} \geq \min(\ln\gamma, \ln\varepsilon) \int_\rho^r f(s)\frac{ds}{s}, \quad r \geq \varepsilon\rho$$
$$\int_r^{+\infty} S_\gamma[f](s)\frac{ds}{s} \geq \ln\gamma \int_r^{+\infty} f(s)\frac{ds}{s}, \quad r \geq \rho.$$

Proposition 3.1.21.

$$S_\delta[q](r) \leq \frac{1+K}{\ln\delta} \int_\rho^r S_\delta[q](s)\frac{ds}{s}, \quad r > \rho\delta,$$

where $K = \sup_{r>\rho} \frac{q(r\delta)}{q(r)}$.

Proof.

$$S_\delta[q](r) = \int_{r/\delta}^{r\delta} q(s)\frac{ds}{s} \leq \int_\rho^r q(s)\frac{ds}{s} + \int_r^{r\delta} q(s)\frac{ds}{s}$$
$$\leq (1+K)\int_\rho^r q(s)\frac{ds}{s} \leq \frac{1+K}{\ln\delta}\int_\rho^r S_\delta[q](s)\frac{ds}{s}.$$

We give the next proposition without proof as it is similar to the one above.

Proposition 3.1.22.

$$S_\delta[q](r) \leq \frac{1+K}{\ln\delta} \int_r^{+\infty} S_\delta[q](s)\frac{ds}{s}, \quad r > \rho\delta,$$

where $K = \sup_{r>\rho} \frac{q(r)}{q(r\delta)}$.

Now we are ready to pass on to the proof of our theorems.

3. Proof of theorems 3.1.1, 3.1.2, 3.1.3, 3.1.4

For a given function V which will be defined separately for each case, we will denote

$$R_1(r) = \int_\rho^r V(s)\frac{ds}{s}, \quad R_2(r) = \int_r^{\rho_1} V(s)\frac{ds}{s},$$
$$R_{2,\rho_1}(r) = \int_r^{\rho_1} V(s)\frac{ds}{s}, \quad I = \int_\rho^\infty V(s)\frac{ds}{s}.$$

In the proof below the letter C stands for constants independent on ξ, ρ_1.

First, we set $V = Q$ in case I, $V = S_\delta[q]$ in case II, $V = \left|\frac{Q^2}{rQ'}\right|$ in case III, $V = Q\ln r$ in case IV. Note that in each case except IV

$$k_1 := \sup_{r>\rho} \frac{V}{c(x)} < +\infty.$$

(In the cases I and III $k_1 = 1$, and in the case II $k_1 \leq 2\ln\delta$). In case IV we set $k_1 = 0$.

In cases labeled by **a.**

$$R_{2,\rho_1}(r) \to +\infty \text{ as } \rho_1 \to \infty,$$
$$R_1(r) \to +\infty \text{ as } r \to \infty.$$

In cases labeled by **b.** $I < +\infty$ and

$$R_{2,\rho_1}(r) \to R_2(r) \text{ as } \rho_1 \to +\infty,$$
$$R_1(r) \to I \text{ as } r \to \infty,$$
$$R_2(r) \to 0 \text{ as } r \to \infty,$$
$$R_1(r),\ R_{2,\rho_1}(r) \leq I \quad \text{for all} \quad r, \rho_1 \geq \rho.$$

Let us fix some number $\xi > 2\rho$. We start with applying Lemma 3.1.19 to the annulus A_{ρ,ρ_1}, $\rho_1 > \xi$. It immediately yields the following estimate

$$M(\xi) \leq \frac{\sigma+1}{\sigma-1}\max\left(C_1(\xi)R_1^{\frac{2}{1-\sigma}}(\xi), C_{2,\rho_1}(\xi)R_{2,\rho_1}^{\frac{2}{1-\sigma}}(\xi)\right), \quad \xi \in (\rho, \rho_1), \quad (3.1.21)$$

where

$$C_1(\xi) = \left[\sup_{x\in A_{\rho,\xi}} \frac{2\Phi}{\sigma-1}\cdot\left(\frac{\sigma+1}{\sigma-1}\frac{V^2}{c(x)} + \left((2-A(x))\frac{V}{c(x)} - \frac{rV'}{c(x)}\right)_+ R_1\right)\right]^{\frac{1}{\sigma-1}}$$

$$C_{2,\rho_1}(\xi) = \left[\sup_{x\in A_{\xi,\rho_1}} \frac{2\Phi}{\sigma-1}\cdot\left(\frac{\sigma+1}{\sigma-1}\frac{V^2}{c(x)} + \left((A(x)-2)\frac{V}{c(x)} + \frac{rV'}{c(x)}\right)_+ R_{2,\rho_1}\right)\right]^{\frac{1}{\sigma-1}}.$$

Let us denote

$$f_1(r) = \sup_{|x|=r} \left|\frac{rV'}{c(x)}\right|, \quad f_2(r) = \sup_{|x|=r} \frac{V^2(r)}{c(x)}.$$

We claim that in each case

$$k_2 := \sup_{r>\rho} f_1(r) < +\infty.$$

Indeed, for case I it is contained in the condition of the theorem, since in this case $V = Q$ and

$$f_1(r) \le \left|\frac{rQ'}{Q}\right|.$$

In case II it follows from the property of $S_\delta[q]$:

$$|\frac{d}{dr}S_\delta[q]| \le \delta\frac{\inf_{|x|=r} c(x)}{r}.$$

In case III we see that

$$f_1(r) \le \left|\frac{rV'}{Q}\right| = \left|2 - \frac{Q}{rQ'} - \frac{QQ''}{(Q')^2}\right|,$$

which is uniformly bounded in $A_{\rho,\infty}$ according to the condition of the theorem. (Without loss of generality we can assume that $|Q'(\rho)| > 0$ since otherwise we could apply our reasoning to slightly smaller annulus in which $|Q'| > const > 0$.)

In case IV we evaluate

$$f_1(r) \le \left|1 + \frac{Q'r\ln r}{Q}\right|,$$

which is again uniformly bounded in $A_{\rho,\infty}$.

Second, we claim that in cases labeled by **a.** there exist $k_3, k_4 > 0$ such that

$$f_2(r) \le k_3 + k_4 R_1(r),$$

and in cases labeled by **b.** there exists a constant $k_4 > 0$ such that

$$f_2(r) \le k_4 R_2(r).$$

In case I we estimate first

$$f_2(r) \le Q(r)$$

and note that $|Q'(r)| \le CQr^{-1}$. Our statement follows then from the Newton-Leibnitz formula.

In case II we estimate first

$$f_2(r) \le 2\ln\delta S_\delta[q](r)$$

and use Propositions 3.1.21 and 3.1.22 to obtain the desired estimate.

In case III we write first

$$f_2(r) \le \frac{Q^3}{r^2(Q')^2}$$

and explicitly calculating the derivative of the last expression we see that

$$\left(\frac{Q^3}{r^2(Q')^2}\right)' = \frac{3Q^2}{r^2Q'} - \frac{2Q^3}{r^3(Q')^2} - \frac{2Q^3Q''}{(Q')^3},$$

which is $\le 3(R_1)'$ in case IIIa. In case IIIb we write it as

$$\frac{Q^2}{r^2Q'}\left(3 - \frac{2Q}{rQ'} - \frac{2QQ''}{(Q')^2}Q'\right),$$

which shows that it is $\ge CR_2'(r)$ with some constant C. (We again assume in this place that $|Q'(\rho)| > 0$).

In case IV

$$f_2(r) \le Q(\ln r)^2$$

and estimating the derivative of the last expression we have

$$\left|\frac{d}{dr}Q(\ln r)^2\right| = \left|Q'(\ln r)^2 + 2\frac{Q\ln r}{r}\right| \le C\frac{Q\ln r}{r} = C(R_1(r))' = -C(R_2(r))'$$

with some constant C. Application of Newton-Leibnitz formula finishes the proof.

Now, in cases labeled as **a.** we estimate

$$\begin{aligned}
C_1(\xi) &\le C(\sigma,\nu_2)\sup_{\rho<\tau<\xi}(k_3 + (k_1+k_2+k_4)R_1(\tau))^{\frac{1}{\sigma-1}} \\
&\le CR_1(\xi), \quad \xi \ge 2\rho, \\
C_{2,\rho_1}(\xi) &\le C(\sigma,\nu_2)\sup_{\xi<\tau<\rho_1}(k_3 + k_4(R_1(\xi) + R_{2,\tau}(\xi)) + (k_1+k_2)R_{2,\rho_1}(\tau))^{\frac{1}{\sigma-1}} \\
&\le C\left(R_1(\xi) + R_{2,\rho_1}(\xi)\right)^{\frac{1}{\sigma-1}}.
\end{aligned}$$

Estimate (3.1.21) now reads as

$$M(\xi) \le C\max\left(R_1^{\frac{1}{1-\sigma}}(\xi), R_{2,\rho_1}^{\frac{2}{1-\sigma}}(\xi)(R_1(\xi) + R_{2,\rho_1}(\xi))^{\frac{1}{\sigma-1}}\right).$$

Sending ρ_1 to ∞ eliminates the second term and finishes the proof.

In case **b.** we estimate

$$\begin{aligned}
C_1(\xi) &\le C(\sigma,\nu_2)\sup_{\rho<\tau<\xi}(k_4R_2(\tau) + (k_1+k_2)R_1(\tau))^{\frac{1}{\sigma-1}} \\
&\le CI^{\frac{1}{\sigma-1}}, \\
C_{2,\rho_1}(\xi) &\le C(\sigma,\nu_2)\sup_{\xi<\tau<\rho_1}(k_4R_2(\tau) + (k_1+k_2)R_{2,\rho_1}(\tau))^{\frac{1}{\sigma-1}} \\
&\le C\left(R_2(\xi) + R_{2,\rho_1}(\xi)\right)^{\frac{1}{\sigma-1}} \le CR_2^{\frac{1}{\sigma-1}}(\xi).
\end{aligned}$$

Estimate (3.1.21) now reads as

$$M(\xi) \le C \max\left(I^{\frac{1}{\sigma-1}} R_1^{\frac{2}{1-\sigma}}(\xi),\ R_2^{\frac{1}{\sigma-1}}(\xi) R_{2,\rho_1}^{\frac{2}{1-\sigma}}(\xi)\right).$$

Sending ρ_1 to ∞ and observing that

$$R_1(\xi) \ge const > 0, \quad \xi \ge 2\rho,$$

finishes the proof. $\square$

4. Results for the punctured ball

In this section we collect the results analogous to those we have obtained for the exterior domain. We do not give proofs here as they repeat those for the preceding case.

Theorem 3.1.23. *Let u be a solution to inequality* (3.1.1) *in $A_{0,\rho}$. Let $Q(\cdot) : (0,\rho) \longrightarrow \mathbb{R}^+$ be such that*

$$Q(r) \le \inf_{|x|=r} c(x),$$

$$\sup_{r<\rho} \left| \frac{Q'r}{Q} \right| < +\infty.$$

a) *If $\int_0^\rho Q(r)\frac{dr}{r} = +\infty$ then*

$$|u(x)| \le C \left(\int_{|x|}^{\rho} Q(r) \frac{dr}{r} \right)^{\frac{1}{1-\sigma}}, \quad x \in A_{0,\rho/2}, \tag{3.1.22}$$

and as a consequence

$$u(x) \longrightarrow 0 \text{ as } x \longrightarrow 0.$$

b) *If $\int_0^\rho Q(r)\frac{dr}{r} < +\infty$ and Q is bounded then*

$$|u(x)| \le C \left(\int_0^{|x|} Q(r) \frac{dr}{r} \right)^{\frac{1}{1-\sigma}}, \quad x \in A_{0,\rho/2}. \tag{3.1.23}$$

Theorem 3.1.24. *Let u be a solution to inequality* (3.1.1) *in $A_{0,\rho}$. Let $q(\cdot) : (0,\rho) \longrightarrow \mathbb{R}^+$ be such that*

$$q(r) \le \inf_{x \in \bar{A}_{r/\delta, r\delta}} c(x) \text{ for some } \delta > 1.$$

a. *If $\int_0^\rho q(r)\frac{dr}{r} = +\infty$ and*

$$\sup_{r<\rho} \frac{q(r/\delta)}{q(r)} < +\infty$$

then

$$|u(x)| \le C \left(\int_{|x|}^{\rho} q(r) \frac{dr}{r} \right)^{\frac{1}{1-\sigma}}, \quad x \in A_{0,\rho/2}. \tag{3.1.24}$$

b. *If* $\int_0^\rho q(r)\frac{dr}{r} < +\infty$, *q is bounded and*

$$\sup_{r<\rho/\delta} \frac{q(r\delta)}{q(r)} < +\infty$$

then

$$|u(x)| \le C \left(\int_0^{|x|} q(r) \frac{dr}{r} \right)^{\frac{1}{1-\sigma}}, \quad x \in A_{0,\rho/2}. \tag{3.1.25}$$

Theorem 3.1.25. *Let u be a solution to inequality* (3.1.1) *in* $A_{\rho,\infty}$. *Let* $Q(\cdot) : (0,\rho) \to \mathbb{R}^+$ *be such that*

$$Q(r) \le \inf_{|x|=r} c(x).$$

a. *Suppose that*

$$\frac{rQ'}{Q} \to -\infty \text{ as } r \to 0 \text{ and}$$

$$Q' \le 0,\ Q'' \ge 0\ ,\ \sup_{r<\rho} \frac{QQ''}{(Q')^2} < +\infty.$$

Then

$$u(x) \to 0 \text{ as } x \to 0$$

and

$$|u(x)| \le C \left(\int_{|x|}^{\rho} \frac{-Q^2}{rQ'} \frac{dr}{r} \right)^{\frac{1}{1-\sigma}}, \quad x \in A_{0,\rho/2}. \tag{3.1.26}$$

b. *Suppose that*

$$\frac{rQ'}{Q} \to +\infty \text{ as } r \to 0 \text{ and}$$

$$Q' > 0,\ Q'' \ge 0\ ,\ \sup_{r<\rho} \frac{QQ''}{(Q')^2} < +\infty.$$

Then

$$|u(x)| \le C \left(\int_0^{|x|} \frac{Q^2}{rQ'} \frac{dr}{r} \right)^{\frac{1}{1-\sigma}}, \quad x \in A_{0,\rho/2}. \tag{3.1.27}$$

Theorem 3.1.26. *Let u be a solution to inequality* (3.1.1) *in* $A_{0,\rho}$, $\rho < 1$. *Let* $Q(\cdot) : (0,\rho) \to \mathbb{R}^+$ *be such that*

$$Q(r) \le \inf_{|x|=r} c(x) \text{ and}$$

$$\sup_{r<\rho} \left| \frac{Q' r \ln r}{Q} \right| < +\infty.$$

Let

$$A(x) \equiv 2.$$

a. *Suppose that*

$$\int_0^\rho Q \ln r \frac{dr}{r} = +\infty.$$

Then

$$|u(x)| \le C \left(\int_{|x|}^\rho Q \ln \frac{1}{r} \frac{dr}{r} \right)^{\frac{1}{1-\sigma}}, \quad |x| < \rho/2. \tag{3.1.28}$$

b. *Suppose that*

$$\int_0^\rho Q \ln r \frac{dr}{r} < +\infty.$$

Then

$$|u(x)| \le C \left(\int_0^{|x|} Q \ln \frac{1}{r} \frac{dr}{r} \right)^{\frac{1}{1-\sigma}}, \quad |x| < \rho/2. \tag{3.1.29}$$

5. Improvement of estimates

Our estimates are aimed at the "worst case" (demonstrated in Example 4) - that is, $A(x) < 2$ in $A_{\rho,\infty}$ or $A(x) > 2$ if we consider $A_{0,\rho}$. If we consider $A_{\rho,\infty}$ (or $A_{0,\rho}$) and

$$\liminf_{x \to \infty} A(x) > 2 \quad \left(\limsup_{x \to 0} A(x) < 2 \text{ resp.} \right)$$

then the estimates provided by Theorems 3.1.1 (3.1.23 resp.) can be improved. That is the reason why in our examples we used the drift term, which shifts the "effective dimension".

Using the expression (3.1.19) for the operator $\mathcal{L}$ acting on radial function one can easily verify that both functions (if they are defined)

$$f_1(r) = \int_r^{+\infty} \exp\left(\int_\rho^s \sup_{|x|=\xi} (2 - A(x)) \frac{d\xi}{\xi} \right) \frac{ds}{s} \tag{3.1.30}$$

$$f_2(r) = \int_0^r \exp\left(\int_s^\rho \sup_{|x|=\xi} (A(x) - 2) \frac{d\xi}{\xi} \right) \frac{ds}{s} \tag{3.1.31}$$

are positive solutions to the inequality $\mathcal{L}u \le 0$ in $A_{\rho,\infty}$ and $A_{0,\rho}$, respectively and hence positive supersolutions to inequality (3.1.1) in the same domain. It is clear that $f_1(r) \to 0$ as $r \to \infty$ and $f_2(r) \to 0$ as $r \to 0$.

The example of f_1 is a fundamental solution to $\triangle u = 0$ in dimension $n \ge 3$, which is $c_n r^{2-n}$. The example of f_2 is a function $f(r) = r$, which gives a solution to $\triangle u - (n-1) \sum_{j=1}^n \frac{x_j u_{x_j}}{|x|^2}$.

Then,

$$f_3(r) = \int_\rho^r \exp\left(\int_\rho^s \inf_{|x|=\xi}(2 - A(x))\frac{d\xi}{\xi}\right)\frac{ds}{s} \tag{3.1.32}$$

$$f_4(r) = \int_r^\rho \exp\left(\int_s^\rho \inf_{|x|=\xi}(A(x) - 2)\frac{d\xi}{\xi}\right)\frac{ds}{s} \tag{3.1.33}$$

are positive solutions to the inequality $\mathcal{L}u \le 0$ in $A_{\rho,\infty}$ and $A_{0,\rho}$ respectively and hence positive supersolutions to inequality (3.1.1) in the same domains. If the integrals defining them diverge then we have a supersolution tending to $+\infty$ as $r \to \infty$ or $r \to 0$.

The example for both f_3, f_4 is provided by the fundamental solution to $\Delta u = 0$ in $\mathbb{R}^2$.

It is easy to see that the function f_1 is defined if $\liminf_{x\to\infty} A(x) > 2$ and the function f_2 is defined if $\limsup_{x\to 0} A(x) < 2$. If, on the other hand, we have $\limsup_{x\to\infty} A(x) < 2$ ($\liminf_{x\to 0} A(x) > 2$) we have a supersolution $f_3(r)$ (resp. $f_4(r)$) which tends to $+\infty$ as x goes to ∞(resp. 0).

To avoid unnecessary "generality" we demonstrate on the simple examples how the estimates can be improved (or established) in several interesting cases. This exposition is restricted for simplicity to the case of the Laplacian. It is readily seen that these results can be easily carried over to general operators of the form (3.1.2), with coefficients which are Dini continuous at 0 or at infinity.

Example 3.1.27. Let u be a (classical) solution to

$$\Delta u = \frac{1}{|x|^2}u^\sigma, \quad x \in A_{\rho,\infty} \subset \mathbb{R}^n,\ n \ge 3.$$

Then Theorem 3.1.1 yields that $u(x) \to 0$ as $x \to \infty$ and using the maximum principle we immediately obtain

$$|u(x)| \le \frac{\sup_{|x|=2\rho}|u(x)|}{(2\rho)^{2-n}}|x|^{2-n}, \quad x \in A_{2\rho,\infty},$$

which is far better that the estimate $|u(x)| \le C\,(\ln|x|)^{\frac{1}{1-\sigma}}$ provided by Theorem 3.1.1.

Example 3.1.28. Let u be a (classical) solution to

$$\Delta u = u^{\frac{n}{n-2}}, \quad x \in A_{0,1} \subset \mathbb{R}^n,\ n \ge 3.$$

The number $\frac{n}{n-2}$ is the critical exponent for the equation of this type. Setting $u = r^{2-n}v$ we have

$$\Delta v + 2(2-n)\sum_{j=1}^n \frac{x_j}{|x|^2}v_{x_j} = \frac{v^{\frac{n}{n-2}}}{|x|^2}.$$

which immediately yields $v(x) \to 0$ as $x \to 0$ and consequently $u(x) = o(|x|^{2-n})$. Using the standard argument one can easily show that this fact implies the "removability of singularity" - we can define $u(0)$ so that the function $u(x)$ will be a solution in the whole ball B_1.

The following example is more subtle because it deals with the equation where the linear potential is present.

Example 3.1.29. Let u be a (classical) solution to

$$\Delta u + \frac{C}{|x|^2} u = |x|^p u^\sigma, \quad x \in A_{1,\infty} \subset \mathbb{R}^n, \ n \geq 3,$$

where the constant $C \leq C_H = \frac{(n-2)^2}{4}$. Let us denote

$$\lambda_+ = \frac{2-n+\sqrt{(n-2)^2-4C}}{2}, \quad \lambda_- = \frac{2-n-\sqrt{(n-2)^2-4C}}{2}.$$

It is clear that $\lambda_+ \geq \lambda_-$ and the only case when they are equal is when $C = \frac{(n-2)^2}{4}$. Now, one can easily check that the functions $r^{\lambda_+}, r^{\lambda_-}$ if $C < \frac{(n-2)^2}{4}$ and $r^{\frac{2-n}{2}}, r^{\frac{2-n}{2}} \ln r$ if $C = \frac{(n-2)^2}{4}$ are positive solutions to

$$\Delta u + \frac{C}{|x|^2} u = 0 \text{ in } A_{1,\infty}.$$

Let λ be λ_+ or λ_-. Performing the "ground-state transform" $u = r^\lambda v$ we arrive at the equation

$$\Delta v + 2\lambda \sum_{j=1}^{n} \frac{x_j}{|x|^2} v_{x_j} = |x|^{p+\lambda(\sigma-1)} v^\sigma.$$

Suppose first that $p + \lambda(\sigma - 1) \neq -2$. In this case Theorem 3.1.1 yields

$$|v(x)| \leq \tilde{C} |x|^{\frac{2+p}{1-\sigma} - \lambda}, \quad |x| > 2,$$

and consequently

$$|u(x)| \leq \tilde{C} |x|^{\frac{2+p}{1-\sigma}}, \quad |x| > 2. \tag{3.1.34}$$

Now, suppose that $p + \lambda(\sigma - 1) = -2$, or, in a more convenient form, $\lambda = \frac{2+p}{1-\sigma}$. In this case the estimate for v provided by Theorem 3.1.1 reads as

$$|v(x)| \leq \tilde{C} \left(\ln |x|\right)^{\frac{1}{1-\sigma}}, \quad |x| > 2,$$

and thus

$$|u(x)| \leq \tilde{C} |x|^{\frac{2+p}{1-\sigma}} \left(\ln |x|\right)^{\frac{1}{1-\sigma}}, \quad |x| > 2. \tag{3.1.35}$$

Thus, if λ_+ or λ_- is equal to $\frac{2+p}{1-\sigma}$ then $u(x) = o(|x|^{\frac{2+p}{1-\sigma}})$ as $x \to \infty$.

3.2 Existence of Positive Solutions of a Semilinear Nondivergence Form Elliptic Equation in a Conical Domain.

This section contains a substantially revised version of my joint paper with Irina Filimonova [39]. While that paper was a short communication with brief sketches of the proofs, here I give detailed proofs. Moreover, in the thesis I used a different construction of comparison functions in the critical case. It allowed me to improve the result. Namely, it is easy to see that $\alpha_- \leq 2-n$ (see notation below), hence any convex monotonically decreasing function satisfies condition (3.2.5). Obviously, the class of functions satisfying condition (3.2.4) is in some sense much narrower.

We study the uniformly elliptic equation

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x)\frac{\partial u}{\partial x_i} = -|u|^\sigma u, \qquad \sigma = const > 0, \qquad (3.2.1)$$

in the cone $K = \{(r,\omega) \in \mathbb{R}^n |\ \omega \in \Omega,\ r > 0\}$, where (r,ω) are the polar coordinates in the space $\mathbb{R}^n$ and Ω is a domain of the class $C^{1,1}$ on the unit sphere in $\mathbb{R}^n$. In the cone $K_R := K \cap \{|x| > R\}$ we study the existence and nonexistence of a positive solution of Eq. (3.2.1) such that

$$u|_{\partial K} = 0. \qquad (3.2.2)$$

It was shown in [47] that the equation $\triangle u + u^{\sigma+1} = 0$ does not have a nontrivial nonnegative solution defined on the entire space $\mathbb{R}^n$ for $0 \leq \sigma < 4/(n-2)$ and has such a solution for $\sigma \geq 4/(n-2)$. For the conical domain, there also exists a critical exponent responsible for the existence of a positive solution of the equation $\triangle + u^{\sigma+1} = 0$ ([5]). Numerous papers deal with the generalization of these results to other elliptic equations. A complete bibliography can be found in monograph [87].

In the present paper, we assume that the coefficients $a_{ij}(x)$ and $a_i(x)$ of Eq. (3.2.1) are bounded and measurable and satisfy the condition

$$a_{ij}(x) - \delta_{ij} = o(1), \quad a_i(x) = o(r^{-1}) \quad \text{as} \quad r \to \infty. \qquad (3.2.3)$$

A solution of Eq. (3.2.1) in the cone K_R with condition (3.2.2) is defined as a function u which belongs to the space $W^{2,n}(K_R \setminus \overline{K_{R_1}})$ for each $R_1 > R$, satisfies the equation almost everywhere, and vanishes on ∂K.

Let λ_1 be the first eigenvalue of the Dirichlet problem for the Beltrami-Laplace operator in Ω, i.e., of the problem $\triangle_\omega \psi + \lambda\psi = 0$, $\psi_{\partial\Omega} = 0$. The eigenvalue λ_1 is simple and the corresponding eigenfunction ϕ can be chosen so as to satisfy the conditions $\min_\Omega \phi = 0$, $\max_\Omega \phi = 1$. By

$$\alpha_- = (1/2)\left(2 - n - \sqrt{(n-2)^2 + 4\lambda_1}\right)$$

we denote the negative root of the equation $\alpha^2 + (n-2)\alpha - \lambda_1 = 0$.

The following assertion is the main result of the present paper.

Theorem 3.2.1. *Let condition* (3.2.3) *be satisfied. The following assertions are valid.*

(i) If $\sigma > -2/\alpha_-$, then there exists R such that in the cone K_R there exists a positive solution of Eq. (3.2.1) *with condition* (3.2.2).

(ii) If $\sigma < -2/\alpha_-$, then for any R the only nonnegative solution of Eq. (3.2.1) *in K_R is trivial.*

(iii) If $\sigma = -2/\alpha_-$ and

$$\operatorname*{ess\,sup}_{|x|=r} \sum_{i,j=1}^{n} |a_{ij}(x) - \delta_{ij}| + |x| \sum_{i=1}^{n} |a_i(x)| \le \gamma(r),$$

where $\gamma(r)$ is a function satisfying the Dini condition $\int^{+\infty} \gamma r^{-1} dr < \infty$ and either the condition

$$\gamma(r) = o(1), \quad \frac{d\gamma}{dr} = o\left(\frac{\gamma(r)}{r}\right), \quad \frac{d^2\gamma}{dr^2} = o\left(\frac{\gamma(r)}{r^2}\right) \quad as \quad r \to \infty \tag{3.2.4}$$

or

$$\frac{d^2\gamma}{dr^2} + \frac{n-1+\alpha_-}{r}\frac{d\gamma}{dr} \ge 0, \tag{3.2.5}$$

then for any R the only nonnegative solution of Eq. (3.2.1) *in K_R is trivial.*

Remark. For the critical exponent $\sigma = -2/\alpha_-$, if $a_{ij}(x) \to \delta_{ij}$ as $r \to \infty$ without any assumptions about the convergence rate, there may exist a positive solution as is illustrated by Example 1. The condition in assertion (iii) in Theorem 3.2.1 is valid, for example, for the functions $\gamma(r) = \ln^{-1-\varepsilon} r$ and $\gamma(r) = \ln^{-1} r \ln^{-1-\varepsilon}(\ln r)$ with $\varepsilon > 0$.

Example 1. Let us show that there exists a function $g(r) \to 0$, $r \to \infty$, such that the equation

$$\sum_{i,j=1}^{n} \left(\delta_{ij} + g(r)\frac{x_i x_j}{r^2}\right) \frac{\partial^2 u}{\partial x_i \partial x_j} + u^{\sigma+1} = 0, \quad \sigma = -\frac{2}{\alpha_-}, \tag{3.2.6}$$

has a positive solution in some cone K_R.

The existence of a positive solution will be proved once we construct a positive supersolution. In the polar coordinates, Eq. (3.2.6) has the form

$$\Delta u + g(r) u_{rr} + u^{\sigma+1} = 0.$$

Since $0 \le \phi \le 1$, it follows that $f(r)\phi$ is the desired solution of Eq. (3.2.6) provided that $f(r)$ is a positive solution of the equation

$$(1+g(r))f'' + \frac{n-1}{r}f' - \frac{\lambda_1}{r^2}f + f^{\sigma+1} = 0,$$

where the prime stands for the derivative with respect to r. One can readily see that the function $f(r) = r^{\alpha_-}\ln^k r$ is a solution of this equation for

$$g(r) = \frac{k(2-n-2\alpha_-)\ln^{-1} r - k(k-1)\ln^{-2} r - \ln^{k\sigma} r}{\alpha_-(\alpha_- - 1) + k(2\alpha_- - 1)\ln^{-1} r + k(k-1)\ln^{-2} r};$$

moreover, $g(r) \to 0$ as $r \to \infty$ if $k < 0$.

Proof of assertion (i) of Theorem 3.2.1. It is known (see, for instance, [3]) that there exists a positive solution of Eq. (3.2.1) satisfying condition (3.2.2) provided there exists a positive supersolution of Eq. (3.2.1). Let a function $\vartheta(\omega)r^{\alpha_-+\varepsilon}$, $\varepsilon > 0$, be a solution of the equation $\Delta u = -r^{-\alpha_- -2+\varepsilon}$ with condition (3.2.2). For the function ϑ, we obtain the equation

$$\Delta_\omega \vartheta + (\alpha_- + \varepsilon)(n - 2 + \alpha_- + \varepsilon)\vartheta = -1.$$

Since $(\alpha_- + \varepsilon)(n-2+\alpha_- + \varepsilon) < \lambda_1$, this equation has a solution. Moreover, the maximum principle is still applicable which implies the positivity of ϑ. By virtue of condition (3.2.3), we conclude that the function $\vartheta(\omega)r^{\alpha_-+\varepsilon}$ is a solution of the equation

$$Lu = f \tag{3.2.7}$$

with the right-hand side f equivalent to $-r^{\alpha_- -2+\varepsilon}$ as $r \to \infty$. The function $w = \vartheta(\omega)r^{\alpha_-+\varepsilon}$ is a supersolution of Eq. (3.2.1) whenever $f + w^{\sigma+1} < 0$. This is the case in some cone K_R provided $-2 > \sigma(\alpha_- + \varepsilon)$. Therefore, for each $\sigma > -2/\alpha_-$ there exists a sufficiently small $\varepsilon(\sigma) > 0$ such that the function $\vartheta(\omega)r^{\alpha_-+\varepsilon}$ is a positive supersolution of Eq. (3.2.1) with condition (3.2.2) for $r > R(\sigma, \varepsilon(\sigma))$.

Once we have constructed a supersolution, the construction of a solution goes as follows. We only outline the proof here, the missing details being quite standard. We assume that in K_R the coefficients of L are sufficiently close to the coefficients of the Laplacian, so all the elliptic estimates and existence results we need hold. For $\rho > R$ define the domains $\mathcal{A}_{R,\rho}$ as follows. First, define the 'caps' $\mathcal{S}_1(\rho) = \{(r,\omega) \in \mathbb{R}^n : r = \rho - \rho\sqrt{\phi(\omega)},\ \omega \in \Omega\}$ and $\mathcal{S}_2(R) = \{(r,\omega) \in \mathbb{R}^n : r = R + R\sqrt{\phi(\omega)},\ \omega \in \Omega\}$. By $\mathcal{S}(R,\rho)$ denote the union of $\mathcal{S}_1(\rho)$, $\mathcal{S}_2(R)$ and $\mathcal{S}_3(\rho, R) = \{(r,\omega) \in \mathbb{R}^n : r \in [\rho, R], \omega \in \bar{\Omega}\}$. By $\mathcal{A}_{R,\rho}$ we denote the domain inside $\mathcal{S}(R,\rho)$. By construction, $\mathcal{A}_{R,\rho}$ is a $C^{1,1}$ domain.

Next, fix some $\rho > R$. Set $u_0 = w$. In the domain $\mathcal{A}_{R,\rho}$ consider the following sequence of problems:

$$Lu_{\rho,j} = -u_{\rho,j-1}^{\sigma+1} \quad \text{in} \quad \mathcal{A}_{R,\rho}, \quad \text{and} \quad u_{\rho,j} = w \quad \text{on} \quad \partial\mathcal{A}_{R,\rho},$$

where $j = 1, 2, 3, \ldots$. It is clear that $0 \le u_1 \le w$. From the maximum principle it follows that $u_1 \ge u_2 \ge u_3 \ge \ldots$. On the other hand, $u_j \ge w_{\rho,1}$, where $w_{\rho,1}$ is a solution to the homogeneous equation $Lw_1 = 0$ in $\mathcal{A}_{R,\rho}$ such that $w_{\rho,1} = w$ on $\partial\mathcal{A}_{R,\rho}$. It is not hard to see that the sequence $\{u_{\rho,j}\}$ converges to a non-zero limit u_ρ which is a solution of $Lu - u^{\sigma+1} = 0$ in $\mathcal{A}_{R,\rho}$ equal to w on the boundary. Moreover, $w_1 \le u_\rho \le w$ in $\mathcal{A}_{R,\rho}$. Next, consider the sequence $\{u_{\rho_j}\}$ with $\rho_j = 2^j$. It follows from the standard theory that there is a subsequence of $\{u_j\}$ which converges to a limit in $W^{2,n}(\mathcal{A}_{R,\rho})$ for any ρ. Clearly, this limit is positive, since it has non-zero boundary values. This limit is the desired solution of equation (3.2.1) in $K_{R+\delta}$. The proof of assertion (i) of Theorem 3.2.1 is completed. □

The example of the equation $\Delta u + u^{\sigma+1} = 0$ was used in [65] to show how sharp estimates for a positive solution of a linear equation allow one to prove the absence of a positive solution of a nonlinear equation. Assertions (ii) and (iii) in Theorem 3.2.1 are proved with the use of these ideas.

Suppose that there exists a positive solution of Eq. (3.2.1) with condition (3.2.2). We need to show that this contradicts the following lemma.

Lemma 3.2.2. *Let the operator L be uniformly elliptic, $Q(x)r^2 \to +\infty$ as $r \to \infty$, and $|a_i|r < M$. Let u be a solution of the equation*

$$Lu + Q(x)u = 0$$

in the cone K_{R_0}. Then u changes sign in any cone $K_R, R > R_0$.

This is a nondivergent case counterpart of Lemma 2.7 in [65]. The proof readily follows from the lemma on a large potential. For a nondivergence equation, such a lemma can be found in [59]. For the convenience of the reader, we provide this lemma here with a proof.

Lemma 3.2.3. *Let the operator L be uniformly elliptic, and the coefficients $|a_i|$ be bounded in the unit ball B_1 in $\mathbb{R}^n$. Then there exists a constant C_0, depending on the constants of ellipticity of L, the supremum of $|a_i|$ and on n, such that any nonnegative supersolution to the equation $Lu + Q(x)u = 0$ in B_1 is identically zero if $Q(x) \ge C_0$.*

Proof. Assume without loss that the ball B_1 is centered at the origin. Now, consider the function $w(x) = (1 - |x|^2)^2$. It is easy to see that in the polar coordinates

$$Lw = 8r^2 \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j + 4(r^2-1)\left(\sum_{i=1}^{n} a_{ii}(x) + \sum_{i=1}^{n} a_i(x)x_i\right) \tag{3.2.8}$$

where we have denoted $\xi_i = x_i r^{-1}$. Now, we have $Lw + Q(x)w > 0$ if we choose $Q(x) \geq C_0$ with C_0 sufficiently large. Near the boundary, the positivity is given by the first term on the right-hand side of (3.2.8). In the remaining part of B_1, the function w is bounded from below by a positive constant, and thus the positivity of $Lw + Qw$ also follows.

Note that due to the strict maximum principle, u is strictly positive inside B_1. Note also that $\frac{\partial w}{\partial n} = 0$ on ∂B_1. On the other hand, the Hopf-Oleinik lemma (see, for instance, [48]) yields $\frac{\partial u}{\partial n} > K \min_{|x|=1/2} u > 0$. Here n denotes the inward-pointed normal and K is a positive constant.

Therefore, $w(x) = o(u(x))$ as x goes to ∂B_1, which yields that there exists a constant A such that $u - Aw \geq 0$ in B_1 and $u(x_0) - Aw(x_0) = 0$. Moreover, the function $u - Aw$ satisfies the inequality $L(u - Aw) + Q(u - Aw) < 0$, which implies $L(u - Aw) < 0$. The last inequality contradicts the maximum principle and the fact that x_0 is the point of minimum for $u - Aw$. □

Lemma 3.2.2 can be applied to a solution of Eq. (3.2.1) with condition (3.2.2) in the conical domain $K'_R = \{(r, \omega \in \mathbb{R}^n : \omega \in \Omega' \Subset \Omega, r > R\}$ provided that we know that $u^\sigma r^2 \to \infty$ in the cone K'_R as $r \to \infty$. The derivation of such formula is the main difficulty in the proof of assertions (ii) and (iii) of Theorem 3.2.1. First, we consider the simpler case (ii).

Proof of assertion (ii) of Theorem 3.2.1. Let $\vartheta(\omega)$ be a solution of the problem

$$\triangle_\omega \vartheta + (\alpha_- - \varepsilon)(\alpha_- - \varepsilon + n - 2)\vartheta = 1, \quad \omega \in \Omega, \quad \vartheta = 0, \quad \omega \in \partial\Omega.$$

The existence of such solution can be obtained from the variational principle or just from the basic properties of eigenvalues. The parameter $\varepsilon > 0$ can be chosen small enough to ensure that $\sigma < -2/(\alpha_- - \varepsilon)$. Since $(\alpha_- - \varepsilon)(\alpha_- - \varepsilon + n - 2) > \lambda_1$, the following antimaximum principle implies that the function $\vartheta(\omega)$ is positive.

Theorem 3.2.4. *Let u be a solution of the problem*

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x)\frac{\partial u}{\partial x_i} + a(x)u + \lambda u = f(x), \quad x \in \Omega,$$
$$u(x) = 0, \quad x \in \partial\Omega,$$

and suppose that $a_{ij} \in C(\bar{\Omega})$, $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j > 0$ for all $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, $a_i, a \in L_\infty(\Omega)$, $f \in L_p(\Omega)$ where $p > n$, and $f > 0$. Let λ_0 be the principal eigenvalue of the homogeneous problem. Then there exists a $\delta > 0$ depending on f and such that the inequalities $\lambda_0 < \lambda < \lambda_0 + \delta$ imply that u is positive in Ω.

The antimaximum principle was proved in Theorem 2 in [22].

The function $w = \vartheta(\omega)r^{\alpha_- - \varepsilon}$, $\varepsilon > 0$, is a solution of the equation $\Delta w = r^{\alpha_- - \varepsilon - 2}$ with condition (3.2.2). For a positive solution u of Eq. (3.2.1) with condition (3.2.2), one can obtain the estimate $u(x) \geq c\vartheta(\omega)r^{\alpha_- - \varepsilon}$ in some cone K_R. Indeed, since $\vartheta \in C^1(\bar{\Omega})$ we have $\left|\frac{\partial\vartheta}{\partial n}\right| \leq C_1$ where $C_1 = C_1(\Omega)$ and n stands for the inward-pointed normal to $\partial\Omega$. On the other hand, using the Hopf-Oleinik lemma for u one can easily see that $\frac{\partial u}{\partial n}(R, \omega) > C_2$, $\omega \in \partial\Omega$, where C_2 is a positive constant. Since in any $\Omega' \Subset \Omega$ both $w(R, \omega)$ and $u(R, \omega)$ are positive and continuous we conclude that there exists such a constant $c > 0$ that $u \geq cw$ on $\{(R, \omega) \in \mathbb{R}^n : \omega \in \Omega\}$. Now, consider the domains $K_{R,\rho}$ with $\rho > R$. Set $\varepsilon(\rho) = c\sup_{r=\rho, \omega\in\Omega} w$. Clearly, $\varepsilon(\rho) \to 0$ as $\rho \to \infty$. The maximum principle immediately implies that $u + \varepsilon(\rho) \geq cw$ in $K_{R,\rho}$. Passing to the limit as $\rho \to \infty$, we obtain that $u \geq cw$ in K_R.

Now assertion (ii) in Theorem 3.2.1 follows from Lemma 3.2.2. □

Proof of assertion (iii) of Theorem 3.2.1. In this case, we need the following lemma, which is interesting in itself. We postpone its proof until the end of the paper.

Lemma 3.2.5. *Let $|a_{ij} - \delta_{ij}| = o(\gamma(r))$ and $|a_i| = o(r^{-1}\gamma(r))$ as $r \to \infty$, where $\gamma(r)$ is a function satisfying the assumptions of assertion (iii) in Theorem 3.2.1. Then Eq. (3.2.7) with $f = 0$ in the cone K_{R_0} has a solution u satisfying condition (3.2.2) and the inequalities*

$$c_1 r^{\alpha_-}\phi(\omega) \leq u(x) \leq c_2 r^{\alpha_-}\phi(\omega), \quad 0 < c_1 < c_2.$$

Let w be a solution to the equation $Lu = 0$ constructed in Lemma 3.2.5. Using the same argument as in the previous case, we immediately obtain the lower bound $u \geq cw$ with some constant $c > 0$. Now, consider u as a supersolution to the equation

$$Lu \leq -(cw)^\sigma u = -\frac{V(\omega)}{r^2}u$$

due to the relation $\sigma = -2/\alpha_-$. Here $V(\omega) = c^\sigma\phi^\sigma(\omega)$. By Λ_1 denote the prime eigenvalue of the operator $\Delta_\omega + V$ on Ω with homogeneous Dirichlet boundary conditions. It is clear that $\Lambda_1 < \lambda_1$. By β_- denote the negative root of the equation $\beta^2 + (n-2)\beta = \Lambda_1$. It is obvious that $\beta_- > \alpha_-$. Let $\varepsilon > 0$ be sufficiently small. Let ϑ be a solution to the problem

$$[\Delta_\omega + V(\omega) + (\beta_- - \varepsilon)(\beta_- - \varepsilon + n - 2)]\,\vartheta = 1 \quad \text{in} \quad \Omega, \quad \vartheta = 0 \quad \text{on} \quad \partial\Omega.$$

Since $(\beta_- - \varepsilon)(\beta_- - \varepsilon + n - 2) > \Lambda_1$, the antimaximum principle implies that ϑ is positive in Ω. Denote $w_1 = r^{\beta_- - \varepsilon}\vartheta$. Choose ε so small that $\beta_- - \varepsilon > \alpha_-$.

Applying to the functions u and w_1 the same comparison argument which we used in the proof of assertion (ii), we obtain the lower bound $u \geq cw_1$ in K_R with some positive constant c. Now, in any cone $K'_R = \{(r,\omega) \in \mathbb{R}^n : r > R,\ \omega \in \Omega' \Subset \Omega\}$ we have $r^2 u^\sigma \to \infty$ as $r \to \infty$. The application of Lemma 3.2.2 completes the proof. □

Proof of Lemma 3.2.5. The proof of the lemma will easily follow once we have a supersolution and a subsolution to $Lu = 0$ with the required behavior. Then we can apply the standard argument, which yields a solution to $Lu = 0$ in the cone, squeezed between the sub- and supersolution.

We will show how to construct a subsolution, the construction of the supersolution being essentially the same. For this, we will use the logarithmic coordinates $r = e^t$. In further calculations the dot will stand for the derivative with respect to t.

Assume that $g(r)$ and $\vartheta(\omega)$ are twice differentiable functions with bounded second derivatives. It is easy to see that in the logarithmic coordinates (t,ω) one has

$$\triangle\left[r^{\alpha_-} g(r)\vartheta(\omega)\right] = r^{\alpha_- -2}\left[\ddot g + (n-2+\alpha_-)\dot g + g\Delta_\omega\vartheta + \alpha_-(n-2+\alpha_-)g\right]$$

and

$$\left|(L-\Delta)\left[r^{\alpha_-} g(r)\vartheta(\omega)\right]\right| \leq C\gamma(r) r^{\alpha_- -2}\left[|g| + |\dot g| + |\ddot g|\right]$$

with some constant $C = C(n,\theta)$.

We will look for a subsolution in the form

$$U_1(x) = r^{\alpha_-}G_1(r)\phi(\omega) + r^{\alpha_-}G_2(r)\vartheta(\omega),$$

where the functions G_1, G_2, θ are to be chosen later.

In the logarithmic coordinates it is easy to see that $LU_1 \geq 0$ holds if

$$\begin{aligned}&\left[\ddot G_1 + (n-2+\alpha_-)\dot G_1\right]\phi + \left[\ddot G_2 + (n-2+\alpha_-)\right]\vartheta + [\Delta + \lambda_1]\,\theta G_2 \\ &\qquad \geq C\gamma(r)\left[|G_1| + \dot G_1| + |\ddot G_1| + |G_2| + |\dot G_2| + |\ddot G_2|\right]. \end{aligned} \tag{3.2.9}$$

We will look for G_1 such that $G_1 \sim 1$ as $r \to \infty$.

Denote $k = 2 - n - \alpha_-$. It is easy to see that $k \geq 0$. Let G_1 be a solution to the equation

$$\ddot G_1 - k\dot G_1 = A\gamma(r)$$

such that $G_1 \to 1$ as $t \to \infty$. Here A is a constant to be defined later. It is easy to see that

$$G_1 = 1 + Ae^{kt} \int_t^{\infty} e^{-k\tau} I(\tau) d\tau,$$

where

$$I(t) = \int_t^{\infty} \gamma(\tau) d\tau = \int_{e^t}^{\infty} \frac{\gamma(\rho)}{\rho} d\rho.$$

Moreover,

$$\dot{G}_1 = A \left(ke^{kt} \int_t^{\infty} e^{-k\tau} I(\tau) d\tau - I(t) \right) = O(I(t)),$$
$$\ddot{G}_1 = O(\max(I(t), \gamma(t))).$$

Take ϑ as a solution to the problem

$$(\Delta_\omega + \lambda_1)\vartheta = 1 - c\phi \quad \text{in} \quad \Omega,$$
$$\vartheta = 0 \quad \text{on} \quad \partial\Omega, \quad \int_\Omega \vartheta\phi d\sigma = 0,$$

where the constant c is such that

$$\int_\Omega (1 - c\phi)\phi d\sigma = 0.$$

It follows from the classical theory that $\theta \in C^{1,\alpha}(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$.
Let $G_2(r) = 2C\gamma(r)$.

Assume that (3.2.4) holds. It easily follows that (3.2.9) holds if the following inequality is true

$$A\gamma\phi + o(\gamma)\vartheta + 2C\gamma(1 - c\phi)$$
$$\geq C\gamma \left[|G_1| + |\dot{G}_1| + |\ddot{G}_1| + 2C\gamma + 2C|\dot{\gamma}| + 2C|\ddot{\gamma}| \right]. \tag{3.2.10}$$

Let $A = 2Cc$. Then (3.2.10) obviously holds in some neighbourhood of infinity. To construct a supersolution, one only has to take $-\vartheta$ instead of ϑ and set $A = -2Cc$.

Assume now that (3.2.5) holds. In this case one has to replace ϑ with $\vartheta + B\phi$, with sufficiently large constant B. In the neighbourhood of $\partial\Omega$ inequality (3.2.9) holds due to the smallness of ϑ, and far from the boundary — due to the fact that $\vartheta + B\phi$ can be made positive in any $\Omega' \Subset \Omega$ by choosing B to be sufficiently large. The same changes as in the previous case yield a supersolution. □

If $\sigma = -2/\alpha_-$, then one can estimate a positive solution u of Eq. (3.2.1) with condition (3.2.2) as $u \geq cr^{\alpha_- + \varepsilon}$ in K'_R for large r, where $\varepsilon,\ c = const > 0$. This implies assertion (iii) in Theorem 3.2.1 and completes the proof of the theorem.

3.3 Asymptotic behaviour of solutions of non-divergence type semilinear elliptic equations in conical domains.

1. Introduction.

In this section we study the asymptotic behaviour of solutions to semilinear elliptic equations of Emden-Fowler type in conical domains. Let $\mathbf{S} = \{x \in \mathbb{R}^n : |x| = 1\}$. Let $\mathcal{D}$ be a C^2–domain on $\mathbf{S}$. The cone with the cross-section $\mathcal{D}$ is $\mathcal{K} := \left\{x \in \mathbb{R}^n \backslash \{0\} : \frac{x}{|x|} \in \mathcal{D}\right\}$. A conical layer is denoted by $\mathcal{K}_{a,b} = \{x \in \mathcal{K} : a < |x| < b\}$. Its lateral boundary is $\Gamma_{a,b} = \{x \in \partial K : a < |x| < b\}$.

In $\mathcal{K}_{R,\infty}$ we consider the problem

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \frac{\sum_{i=1}^{n} b_i(x) u_{x_i}}{|x|} + \frac{c(x)}{|x|^2} u = G(x,u), \tag{3.3.1}$$

$$u|_{\Gamma_{R,\infty}} = 0. \tag{3.3.2}$$

The coefficients a_{ij}, b_i, c are bounded measurable real-valued functions, the matrix a_{ij} is symmetric and satisfies the uniform ellipticity condition: there exist constants $\nu_1, \nu_2 > 0$ such that for $x \in \mathcal{K}_{R,\infty}$, $y \in \mathbb{R}^n$

$$\nu_1 |y|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) y_i y_j \leq \nu_2 |y|^2.$$

The function $G(x,u)$ is measurable in its arguments, continuous in u for almost all x, and satisfies

$$|G(x,u)| \leq C|x|^p |u|^\sigma,$$
$$\text{where} \quad p, \sigma = const, \ \sigma > 1.$$

As a solution to (3.3.1) with boundary condition (3.3.2) in $\mathcal{K}_{R,\infty}$ we understand a function $u \in W^{2,n}_{loc}(\overline{\mathcal{K}}_{R,\infty})$, which satisfies (3.3.1) almost everywhere in $\mathcal{K}_{R,\infty}$ and (3.3.2) almost everywhere on $\Gamma_{R,\infty}$.

We assume that the coefficients of L satisfy the stabilization condition

$$\sum_{i,j=1}^{n} |a_{ij}(x) - a^0_{ij}(\omega)| + \sum_{i=1}^{n} |b_i(x) - b^0_i(\omega)| + |c(x) - c^0(\omega)| \leq C|x|^{-\alpha}, \tag{3.3.3}$$

where $\alpha = const > 0$, $C = const \geq 0$, $\omega = \frac{x}{|x|}$. The coefficients $a^0_{ij} \in C(\overline{\mathcal{D}})$ and b^0_i, c^0 are bounded measurable real-valued functions on $\mathcal{D}$. Under these assumptions we give the complete description of solutions to (3.3.1),(3.3.2) which in some neighbourhood of infinity satisfy the bound

$$|u(x)| \leq C|x|^{\frac{2+p}{1-\sigma}-\varepsilon}. \tag{3.3.4}$$

with some ε, $C > 0$. It turns out that asymptotically such solutions are always close to the linear combinations of special solutions of $Lu = 0$. Under an additional natural assumption on $G(x, u)$ we also prove the existence of a solution with a prescribed asymptotical behaviour of this type.

To state the theorems we need to introduce some further notation. Let

$$L_0 = \sum_{i,j=1}^{n} a_{ij}^0(\omega)\partial_{x_i}\partial_{x_j} + \frac{\sum_{i=1}^n b_i^0(\omega)\partial_{x_i}}{|x|} + \frac{c^0(\omega)}{|x|^2}.$$

Let us denote

$$\frac{\partial}{\partial x_i} = \xi_i \frac{\partial}{\partial r} + \frac{\Omega_i}{r},$$

where $\xi_i = \frac{x_i}{r}$, and Ω_i is a first order differential operator on $\mathbf{S}$. It is easy to check the following properties of Ω_i :

$$\Omega_i \xi_j = \delta_{ij} - \xi_i \xi_j, \sum_{i=1}^n \Omega_i \xi_i = n-1; \ \sum_{i=1}^n \Omega_i^2 = \triangle_\theta.$$

Straightforward computation shows that in polar coordinates L_0 acts as

$$\begin{aligned} L_0 = \sum_{i,j=1}^n a_{ij}^0 \xi_i \xi_j \partial_{rr} &+ \left(\sum_{i=1}^n (a_{ii}^0 + b_i^0 \xi_i) - \sum_{i,j=1}^n a_{ij}^0 \right) \frac{\partial_r}{r} \\ &+ \sum_{i,j=1}^n a_{ij}^0 (\xi_j \Omega_i + \xi_i \Omega_j) \frac{\partial_r}{r} \\ &+ r^{-2} \left[\sum_{i=1}^n b_i^0 \Omega_i + \sum_{i,j=1}^n a_{ij}^0 \Omega_i \Omega_j + c^0 \right]. \end{aligned}$$

Let us denote

$$\Phi^0 = \sum_{i,j=1}^n a_{ij}^0 \xi_i \xi_j, \ T^0 = \sum_{i=1}^n (a_{ii}^0 + b_i^0 \xi_i).$$

The logarithmic change of variables $r = e^t$ transforms the equation $L_0 u = f$ to

$$\begin{aligned} L_1 u := \Bigg[\Phi^0 \partial_{tt} + (T^0 - 2\Phi^0)\partial_t &+ \sum_{i,j=1}^n a_{ij}^0 (\xi_i \Omega_j + \xi_j \Omega_i)\partial_t + \sum_{i=1}^n b_i^0 \Omega_i \\ + \sum_{i,j=1}^n a_{ij}^0 \Omega_i \Omega_j + c^0 \Bigg] & u(t, \omega) = e^{2t} f(t, \omega) := F(t). \end{aligned} \tag{3.3.5}$$

In the Fourier images (with respect to t) the operator in the left-hand side of the last expression acts as

$$\mathcal{L}(\lambda) \equiv \sum_{i,j=1}^{n} a_{ij}^0 \Omega_i \Omega_j + \left(\sum_{i=1}^{n} b_i^0 \Omega_i + \sum_{i,j=1}^{n} a_{ij}^0 (\xi_i \Omega_j + \xi_j \Omega_i)(i\lambda) \right)$$
$$+ \left(\Phi^0 (i\lambda)^2 + (T^0 - 2\Phi^0)(i\lambda) + c^0 \right).$$

It is clear that the condition $u|_{\partial\mathcal{K}} = 0$ becomes

$$\tilde{u}(\lambda, \cdot)|_{\partial\mathcal{D}} = 0.$$

Let us consider the following problem with parameter

$$\mathcal{L}(\lambda) w(\lambda) = f \in L^2(\mathcal{D}),\ w(\lambda)|_{\partial\mathcal{D}} = 0. \tag{3.3.6}$$

We denote $\mathring{X} = W^{2,2}(\mathcal{D}) \cap \mathring{W}^{1,2}(\mathcal{D})$.

From the standard Fredholm theory of elliptic equations it follows that (3.3.6) is uniquely solvable for all $\lambda \in \mathbb{C}\backslash T_L$, where T_L is a countable set without finite limit points. In other words, if $\lambda \in \mathbb{C}\backslash T_L$, then $\operatorname{Ran} \mathcal{L}(\lambda) = L^2(\mathcal{D})$ and $\operatorname{Ker} \mathcal{L}(\lambda) = 0$, if we consider $\mathcal{L}(\lambda)$ on $\mathring{X}$. Any point λ of the "exceptional" set T_L is a pole of some order for the resolvent of (3.3.6). We denote this order by $m(\lambda)$. In the neighbourhood of $\lambda_0 \in T_L$ the resolvent of (3.3.6) admits the decomposition

$$\mathcal{R}(\lambda) f = \sum_{s=1}^{m(\lambda)} B_s(\lambda_0)(\lambda - \lambda_0)^{-s} f + \Gamma(\lambda) f, \tag{3.3.7}$$

where operators $B_s(\lambda_0)$, $s = 1, \dots, m(\lambda_0)$ have finite range. The operator $B_{m(\lambda)}$ is a projection on the kernel of (3.3.6) for $\lambda = \lambda_0$, and $B_j, j = 1, \dots, m(\lambda_0) - 1$ are the projections on the spaces of root functions of the problem (3.3.6). The operator-valued function $\Gamma(\lambda)$ is holomorphic in the neghbourhood of λ_0.

For $\lambda \in T_L$ by $Y(\lambda)$ we denote the set of functions $w(t, \theta) = \underset{\lambda_0}{Res}\, e^{i\lambda t} \mathcal{R}(\lambda) F(\lambda)$, where $F(\lambda)$ runs over the whole set of functions with values in $L^2(\mathcal{D})$ which are holomorphic in some neighbourhood of λ. It is easy to see that $Y(\lambda)$ is a finite-dimensional space, and in the polar coordinates each function $f \in Y(\lambda)$ is

$$f(r, \theta) = r^{i\lambda} \sum_{k=1}^{dimY(\lambda)} c_k \sum_{s=1}^{m(\lambda)} (\ln r)^{s-1}\, \Psi_{\lambda,k,s}(\theta), \tag{3.3.8}$$

where $\Psi_{\lambda,k,s}(\theta) \in \mathring{X}$.

One can easily verify that $Y(\lambda) \subset Ker\, L_0$. Indeed, the operation of taking the residue acts with respect to λ, whereas the operator L_0 acts in variables

(t, ω). Therefore,

$$L_0 \operatorname*{Res}_{\lambda} e^{i\lambda t}\mathcal{R}(\lambda)\tilde{F}(\lambda) = \operatorname*{Res}_{\lambda} L_0 e^{i\lambda t}\mathcal{R}(\lambda)\tilde{F}(\lambda)$$
$$= \operatorname*{Res}_{\lambda} e^{i\lambda t}\mathcal{L}(\lambda)\mathcal{R}(\lambda)\tilde{F}(\lambda) = \operatorname*{Res}_{\lambda} e^{i\lambda t}\tilde{F}(\lambda) = 0,$$

since the function under the last residue is holomorphic. It is clear that all the elements of $Y(\lambda)$ satisfy the homogenous boundary condition (3.3.2).

Note that for each $\lambda \in T_L$ we have $-\overline{\lambda} \in T_L$ and $f \in Y(\lambda)$ iff $\overline{f} \in Y(-\overline{\lambda})$. It is clear that $m(\lambda) = m(-\overline{\lambda})$ and the dimensions of the root space for $\mathcal{L}(\lambda)$ and $\mathcal{L}(-\overline{\lambda})$ coincide.

Throughout this article we use the following notation. For every $f \in Y(\lambda)$ in some neighbourhood of infinity we can construct $\mathcal{P}[f]$ which solves $Lu = 0$, is zero on $\partial\mathcal{K}$, and satisfies

$$\mathcal{P}[f] - f = O(r^{i\lambda-\alpha+\varepsilon})$$

for every $\varepsilon > 0$.

In $\mathcal{K}_{0,R}$, where the coefficients of L are not defined, we set them to be equal to the coefficients of L_0.

By L_R we denote the operator whose coefficients coincide with the coefficients of L for $|x| > R$ and with the coefficients of L_0 for $|x| \leq R$.

By $\theta_R(x)$ we denote the smooth function such that $\theta_R(x) = 0$ if $|x| < R$, $\theta_R(x) = 1$ if $|x| > 2R$, $0 \leq \theta_R(x) \leq 1$, $|\nabla\theta_R(x)| \leq 4R^{-1}$, $|\nabla^2\theta_R(x)| \leq 4R^{-2}$.

$\Pi(\lambda_0)$ denotes the horizontal strip on the complex plane,

$$\Pi(\lambda_0) := \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda_0 \leq \operatorname{Im}\lambda < \sigma \operatorname{Im}\lambda_0 - 2 - p\}.$$

Theorem 3.3.1. *Let u be a solution of (3.3.1), (3.3.2) in $\mathcal{K}_{R,\infty}$. Assume that $u = O(|x|^\beta)$ as $x \to \infty$. Let*

$$\beta\sigma + p + 2 < \beta.$$

Then there exists a number $\lambda_0 \in T_L$, such that

$$-\operatorname{Im}\lambda_0\sigma + p + 2 < -\operatorname{Im}\lambda_0, \tag{3.3.9}$$

and in the neighbourhood of infinity there holds

$$u(x) = \sum_{\lambda} \mathcal{P}[f_\lambda] + O(|x|^{-\operatorname{Im}\lambda_0\sigma+p+2+\varepsilon}) \tag{3.3.10}$$

where ε is an arbitrarily small positive number, $f_\lambda \in Y(\lambda)$, and the summation is over $\lambda \in \Pi(\lambda_0)$.

Theorem 3.3.2. *Let $G(x,u)$ satisfy the additional assumption*

$$|G(x,u) - G(x,v)| \le C|x|^p \left(|u| + |v|\right)^{\sigma-1} |u-v|, \tag{3.3.11}$$

for some $C > 0$. Fix $\lambda_0 \in T_L$ satisfying (3.3.9). *For all $\lambda \in T_L \cap \Pi(\lambda_0)$ take $f_\lambda \in Y(\lambda)$. Then there exists R, which depends on the given data, such that in $\mathcal{K}_{R,\infty}$ there exists a solution of* (3.3.1), (3.3.2) *with asymptotics* (3.3.10).

Remark 3.3.3. The "given data" in the statement of the Theorem 3.3.2 stands for the set of the coefficients of L_0, domain $\mathcal{D}$, the parameters C, α in (3.3.3), n, p, σ, λ_0, and the choice of f_λ. From the proof it is clear that if we take f_λ large, then R provided by Theorem 3.3.2 is also large. This is in the essence of the problem.

The questions of describing the asymptotic behavior of the solutions of equations of the type $\Delta u = \pm|u|^{\sigma-1}u$ attracted a great deal of attention from many authors over the last several decades. The reader is referred to [117], [118] for a relatively full account of the results in this field. In the same monographs one can find many useful references on the subject. Great variety of results on a priori estimates and nonexistense of solutions can be found in [87]. We also mention a recent review [68] where questions of existence and nonexistence of solutions to semilinear elliptic equations in exterior domains were studied.

Let us now give one example of the case when all the solutions fall into the class described in this work. Consider $\Delta u = |u|^{\sigma-1}u$ with $\sigma > \frac{n}{n-2}$. All solutions of this equation in the neighbourhood of infinity have the asymptotics of the type considered here ([103],[104]). We also note that for the equation $\Delta u = \pm|u|^{\sigma-1}u$ in a neighbourhood of a point or in a neighbourhood of a boundary singularity such phenomenon was known for a long time ([49, 116, 117]).

This work stems from the following simple observation: if condition (3.3.4) is satisfied then

$$Lu = Vu, \quad \text{where} \quad V = O(|x|^{-2-\delta})$$

for some $\delta > 0$. For second-order elliptic equations it is known that the perturbation of this order at infinity is "weak" in comparison with the main part. The idea to introduce the "perturbed" solutions $\mathcal{P}[f]$ to describe the "tail" of the asymptotical expansion was taken from the work [88]. The technical part of this paper is based on theory of the weighted spaces of V.A. Kondratiev [62]. In the author's opinion, the advantage of this technique is that it provides the unified approach to the wide class of problems. Moreover, if the exact nonlinearity is given, the same approach allows one to obtain immediately further terms of the asymptotic expansion.

A priori estimate (3.3.4) can be obtained, for example, by the method of barriers ([61, 105]) or by the method given in [34, 97, 63]. To exemplify it, let us note that for the equation $\Delta u = |u|^{\frac{2}{n-2}}u$ in the exterior of the unit ball the latter method gives the following alternative. Either the solution is of constant sign in some neighbourhood of infinity and has the asymptotics described in [115], or it is signchanging in any neighbourhood of infinity and has the asymptotics described in Theorem 3.3.1. It is also worth mentioning an extremely powerful method of obtaining a priori estimates developed by A.A. Kon'kov (see [71] and references therein).

2. Auxiliary results.

The proof relies on two well-known statements about the solutions of linear equations. By $\mathcal{H}_a^k$ we denote the closure of $C^\infty(\overline{\mathcal{K}}\backslash\{0\})$ with respect to the norm

$$\|u\|^2_{\mathcal{H}_a^k} = \int_{\mathcal{K}} \sum_{s=0}^{k} |D^s u|^2 r^{a+2s-2k}\, dx.$$

By $\overset{\circ}{\mathcal{H}}{}_a^2$ we denote the closed subspace of $\mathcal{H}_a^2$ which consists of function with zero trace on the boundary ∂K.

Lemma 3.3.4. *Let $a \in \mathbb{R}$ be such that the line $\operatorname{Im}\lambda = h_1 := \frac{a+n-4}{2}$ does not cross T_L. Then for any $f \in \mathcal{H}_a^0$ the problem $L_0 u = f$ has the unique solution $u \in \overset{\circ}{\mathcal{H}}{}_a^2$. Moreover, $\|u\|_{H_a^2} \le C\|f\|_{H_a^0}$, with $C = C(n, a, L_0)$.*

Lemma 3.3.5. *Let $u \in \overset{\circ}{\mathcal{H}}{}_{a_1}^2$ be a solution to $L_0 u = f$ in $\mathcal{K}$. Let also $f \in \mathcal{H}_{a_2}^0, a_2 > a_1$. Let the lines $\operatorname{Im}\lambda = h_j := \frac{a_j+n-4}{2}, j = 1, 2,$ do not cross T_L. Then*

$$u = \sum_{\lambda} f_\lambda + u_1 \tag{3.3.12}$$

where $f_\lambda \in Y(\lambda)$, $u_1 \in \overset{\circ}{\mathcal{H}}{}_{a_2}^2$, and the sum is taken over all $\lambda \in T_L$ lying in the strip

$$\frac{a_1+n-4}{2} < \operatorname{Im}\lambda < \frac{a_2+n-4}{2}.$$

The numbers c_k in decomposition (3.3.8) for f_λ and $\|u_1\|_{\mathcal{H}_{a_2}^2}$ are estimated from above by $C(\|u\|_{\mathcal{H}_{a_1}^2} + \|f\|_{\mathcal{H}_{a_2}^0})$ where $C = C(n, L_0, a_1, a_2)$.

We only briefly outline the proof in generality required here. Missing details can be found in [62]. We also refer the reader to the recent books [75, 76] where the detailed exposition of the subject and properties of corresponding operator pencils can be found together with the extensive collection of references.

In the proofs of these lemmas we use the "logarithmic" coordinates $x = x(t, \omega)$, $t \in \mathbb{R}$, $\omega \in \mathcal{D}$. Here $t = \ln|x|$ and $\omega = \frac{x}{|x|}$.

Proof of Lemma 3.3.4. Let $v \in \mathcal{H}^k_a$. It is easy to see that in the logarithmic coordinates

$$C_1\|v\|_{\mathcal{H}^k_a} \le \int_{\mathbb{R}}\int_{\mathcal{D}} \sum_{\alpha+\beta\le k} \left|\partial_t^\alpha \partial_\omega^\beta v\right|^2 e^{(a+n-2k)t}\, dt d\omega \le C_2\|v\|_{\mathcal{H}^k_a},$$

where C_1, C_2 are independent of v. Let $\tilde{v}(\lambda,\omega)$ be the Fourier image of $v(t,\omega)$. It is defined on the line $\operatorname{Im}\lambda = h_1 := \frac{a+n-2k}{2}$, and

$$C_3\|v\|_{\mathcal{H}^k_a} \le \int_{\operatorname{Im}\lambda=h_1}\int_{\mathcal{D}} \sum_{s=0}^{k}(1+|\lambda|)^{2s}|\nabla_\theta^{k-s}\tilde{v}|^2\, d\lambda d\theta \le C_4\|v\|_{\mathcal{H}^k_a}. \tag{3.3.13}$$

Under the action of the Fourier transform w.r.t. to t the equation $L_0 u = f$, written in the form (3.3.5), becomes

$$\mathcal{L}(\lambda)\tilde{u}(\lambda,\omega) = \tilde{F}(\lambda,\omega), \quad \tilde{u}(\lambda)|_{\partial\mathcal{D}} = 0,$$

where $\tilde{F}$ is the Fourier image of $e^{2t}f(t,\omega)$. Since the line $\operatorname{Im}\lambda = h_1$ does not cross T_L, we can define $\tilde{u}(\lambda,\omega) = \mathcal{R}(\lambda)\tilde{F}(\lambda,\omega)$ for λ on this line. Moreover, for all λ lying on this line the following well-known estimate holds

$$\sum_{s=0}^{2}(1+|\lambda|)^{2s}\|\tilde{u}(\lambda,\omega)\|^2_{W^{2-s,2}(\mathcal{D})} \le C_5\|\tilde{F}(\lambda)\|^2_{L^2(\mathcal{D})}.$$

Let us denote by $u(t,\omega)$ the inverse Fourier transform of $\tilde{u}(\lambda,\omega)$. We integrate the last inequality over the line $\operatorname{Im}\lambda = h_1$ and use the relation (3.3.13) for both sides to obtain

$$\|u\|_{\mathring{\mathcal{H}}^2_a} \le C_6\|f\|_{\mathcal{H}^0_a}.$$

From the properties of the Fourier transform it follows that $u \in W^{2,2}_{loc}(\overline{\mathcal{K}}\backslash\{0\})$, and is a solution to the original equation.

Proof of Lemma 3.3.5. It is clear that $L_0 u = f \in \mathcal{H}^0_{a_1} \cap \mathcal{H}^0_{a_2}$. Hence,

$$f \in \mathcal{H}^0_a \quad \text{for all} \quad a \in (a_1, a_2). \tag{3.3.14}$$

(this follows easily from the Cauchy-Schwarz inequality). Let $\tilde{F}(\lambda)$ be the Fourier transform of $e^{2t}f(t,\omega)$ w.r.t. t. We consider $\tilde{F}(\lambda)$ as a function of λ with the values in $L^2(\mathcal{D})$. From (3.3.14) it follows that $\tilde{F}(\lambda)$ is holomorphic in the strip $h_1 < \operatorname{Im}\lambda < h_2$. Moreover, for all $h \in (h_1, h_2)$

$$\int_{\operatorname{Im}\lambda=h}\|\tilde{F}\|^2_{L^2(\mathcal{D})}\, d\lambda \le C\left(\int_{\operatorname{Im}\lambda=h_1}\|\tilde{F}\|^2_{L^2(\mathcal{D})}\, d\lambda + \int_{\operatorname{Im}\lambda=h_2}\|\tilde{F}\|^2_{L^2(\mathcal{D})}\, d\lambda\right).$$

From the absence of the poles of $\mathcal{R}(\lambda)$ on the line $\operatorname{Im}\lambda = h_1$ it follows that $\tilde{u}(\lambda) = \mathcal{R}(\lambda)\tilde{F}(\lambda)$. The right-hand side of the last expression is meromorphic in the strip $h_1 < \operatorname{Im}\lambda < h_2$ as a composition of the meromorphic operator–valued function $\mathcal{R}(\lambda)$ with the poles in points of T_L and holomorphic function $\tilde{F}(\lambda)$. Let $u_1 \in \overset{\circ}{\mathcal{H}}{}^2_{a_2}$ be a solution of $L_0 u_1 = f$ constructed in Lemma 3.3.4. Let $\tilde{u}_1(\lambda, \omega)$ be the Fourier image of $u_1(t, \omega)$. It is clear that

$$u_1(t,\omega) = \int_{\operatorname{Im}\lambda = h_2} e^{i\lambda t}\mathcal{R}(\lambda)\tilde{F}(\lambda)\, d\lambda.$$

Utilizing the analogous formula for u and applying the Cauchy residual theorem we obtain

$$\begin{aligned} u(t,\omega) &= \int_{\operatorname{Im}\lambda = h_1} e^{i\lambda t}\tilde{u}\, d\lambda = \int_{\operatorname{Im}\lambda = h_1} e^{i\lambda t}\mathcal{R}(\lambda)\tilde{F}(\lambda)\, d\lambda \\ &= \int_{\operatorname{Im}\lambda = h_2} e^{i\lambda t}\mathcal{R}(\lambda)\tilde{F}(\lambda)\, d\lambda + \sum_{\lambda} \underset{\lambda}{Res}\, e^{i\lambda t}\mathcal{R}(\lambda)\tilde{F}(\lambda) \\ &= u_1(t,\omega) + \sum_{\lambda} \underset{\lambda}{Res}\, e^{i\lambda t}\mathcal{R}(\lambda)\tilde{F}(\lambda), \end{aligned}$$

where the summation is over $\lambda \in T_L$ lying in the strip $h_1 < \operatorname{Im}\lambda < h_2$. After some easy computation the resolvent decomposition (3.3.7) yields that for $\lambda \in T_L$

$$\underset{\lambda}{Res}\, e^{i\lambda t}\mathcal{R}(\lambda)\tilde{F}(\lambda) = e^{i\lambda t}\sum_{s=1}^{m(\lambda)} \frac{(it)^{s-1}}{(s-1)!}\sum_{k=s}^{m(\lambda)} B_k(\lambda)\tilde{F}_{k-s},$$

where $\tilde{F}_j$ are the coefficients of the Cauchy series for holomorphic function $\tilde{F}$ in the neighbourhood of λ:

$$\tilde{F}(z) = \sum_{j=0}^{\infty} \tilde{F}_j (z-\lambda)^j.$$

In the original coordinates we immediately obtain the statement of the lemma.

□

Before proceeding to the auxiliary statements let us make two remarks. First, all the functions which are defined on $\mathcal{K}$ or on its subset are supposed to be extended by zero outside $\mathcal{K}$. Second, when we say that $u = 0$ on the part of the boundary we understand that in the sense of trace.

Lemma 3.3.6. *Let $u \in W^{2,n}_{loc}(\overline{\mathcal{K}}\backslash\{0\})$. Let $u = 0$ on $\partial\mathcal{K}$. Then there exists $C > 0$ such that*

$$|u(y)| \le C\left[|y|^{\frac{-n+4-a}{2}}\|u\|_{\mathcal{H}^0_{a-4}} + |y|^2 \sup_{x\in B_{|y|/2}(y)\cap\mathcal{K}} |Lu(x)|\right].$$

Proof. Naturally, we assume that $\|u\|_{\mathcal{H}^0_{a-4}} < \infty$ (otherwise there is nothing to prove). First, let us recall the well-known a priori estimate (for instance, [48, Theorem 9.21]),

$$\sup_{B_R(y)} |u| \le C \left[\left(\frac{1}{|B_{2R}|} \int_{B_{2R}(y)} |u|^2 dx \right)^{1/2} + R^2 \sup_{x \in B_{2R}(y)} |Lu(x)| \right],$$

where $C = C(n, \nu_1, \nu_2)$. Let $R = |y|/4$. We estimate the integral norm of u as

$$\left(\frac{1}{|B_{|y|/2}|} \int_{B_{|y|/2}(y)\cap\mathcal{K}} |u|^2 \, dx \right)^{1/2}$$

$$\le C|y|^{-n/2} \left(|y|^{4-a} \int_{B_{|y|/2}(y)\cap\mathcal{K}} |u|^2 |x|^{a-4} \, dx \right)^{1/2} \le C|y|^{\frac{4-a-n}{2}} \|u\|_{\mathcal{H}^0_{a-4}}.$$

In the last passage the integral over the part of the cone $B_{|y|/2}(y) \cap \mathcal{K}$ was replaced by the integral over the whole cone $\mathcal{K}$. □

The next lemma is provided in [48, Lemma 9-16] for the continuous coefficients. In fact, the continuity of coefficients is not necessary. It can be easily traced following the proof. It is clear that condition (3.3.15) is satisfied if $a_{ij}(x)$ are sufficiently close to continuous functions. In our case it is so if $\Omega \subset \mathcal{K}_{R,\infty}$ with R large enough.

Lemma 3.3.7. *Let $u \in W^{2,q_1}_{loc}(\Omega) \cap L^{q_1}(\Omega), 1 < q_1 < \infty$ and $Lu \in L^{q_2}(\Omega), q_2 \in (q_1, \infty)$. Let u be zero on the $C^{1,1}$ part of the boundary $T \subset \partial\Omega$. Let*

$$\sup_{x,y\in\Omega, |x-y|\le\delta} |a_{ij}(x) - a_{ij}(y)| < \varepsilon < \varepsilon_1(n, \nu, q_1, q_2) \tag{3.3.15}$$

for some $\delta > 0$. Then $u \in W^{2,q_2}_{loc}(\Omega \cup T)$, and the estimate

$$||u||_{W^{2,q_2}(\Omega')} \le C \left(||u||_{L^{q_1}(\Omega)} + ||Lu||_{L^{q_2}(\Omega)} \right)$$

holds for any $\Omega' \Subset \Omega \cup T$.

Lemma 3.3.8. *Let R_0 be large enough. Let $u \in W^{2,2}_{loc}(\overline{\mathcal{K}} \backslash \{0\})$. Let $u(x) = 0$ if $|x| < 2R_0$ and $u \mid_{\partial\mathcal{K}} = 0$. Then*

$$\|u\|_{\mathring{\mathcal{H}}^2_a} \le C \left(\|u\|_{\mathcal{H}^0_{a-4}} + \|Lu\|_{\mathcal{H}^0_a} \right),$$

where $C = C(n, a, \nu_1, \nu_2)$.

Proof. If either $\|u\|_{\mathcal{H}^0_{a-4}}$ or $\|Lu\|_{\mathcal{H}^0_a}$ is infinite the assertion is trivial. Suppose now that both $\|u\|_{\mathcal{H}^0_{a-4}}, \|Lu\|_{\mathcal{H}^0_a} < \infty$. It suffices to show that for any $j \in \mathbb{N}$

$$\int_{\mathcal{K}_{2^j R_0, 2^{j+1} R_0}} |\nabla u|^2 r^{a-2} + |\nabla^2 u|^2 r^a dx \le C \int_{\mathcal{K}_{2^{j-1} R_0, 5 2^{j-1} R_0}} u^2 r^{a-4} + |Lu|^2 r^a dx \tag{3.3.16}$$

with the constant C independent of j and u. The summation of this inequality over all j yields the result of the lemma. Perform the change of variables $x = R_0 2^j y$. Then the last inequality is equivalent to

$$\int_{\mathcal{K}_{1,2}} |\nabla \tilde{u}_j|^2 + |\nabla^2 \tilde{u}_j|^2 dy \leq C_1 \int_{\mathcal{K}_{1/2,5/2}} |\tilde{u}_j|^2 + |\tilde{L}_j \tilde{u}_j|^2 dy, \tag{3.3.17}$$

where $\tilde{u}_j(y) = u(R_0 2^j y)$, and

$$\tilde{L}_j \tilde{u} = \sum_{i,l=1}^{n} a_{il}(x) \tilde{u}_{y_i y_l} + \sum_{i=1}^{n} b_i(x) \tilde{u}_{y_i} + c(x)\tilde{u}.$$

In virtue of (3.3.3), by the appropriate choice of R_0 we can ensure that for all $j \in \mathbb{N}$ the coefficients of $\tilde{L}_j$ are sufficiently close in $\mathcal{K}_{1/2,5/2}$ to the continuous functions $a^0_{ij}(\frac{y}{|y|}), b^0_i(\frac{y}{|y|}), c(\frac{y}{|y|})$. Now the scheme of the proof of the estimate (3.3.17), provided in [48] (Theorem 9–11) for the operator with continuous coefficients goes without changes. □

From Lemma 3.3.8 we immediately obtain

Proposition 3.3.9. *Let R be large enough. Let $v \in W^{2,2}_{loc}(\overline{\mathcal{K}}_{R,\infty})$ and $v\,|_{\partial\mathcal{K}} = 0$. Suppose also that, $v(x) \equiv 0$ for $|x| < R$, $|v(x)| \leq C|x|^{b+2}$, $|Lv(x)| \leq C|x|^b$. Then $v \in \overset{\circ}{\mathcal{H}}{}^2_a$ for all $a < -n - 2b$.*

Proposition 3.3.10. *Let $a \in \mathbb{R}$ be such that the line $\operatorname{Im}\lambda = \frac{a+n-4}{2}$ does not cross T_L. Then for all sufficiently large R the operator L_R is a bounded invertible operator from $\overset{\circ}{\mathcal{H}}{}^2_a$ to $\mathcal{H}^0_a$.*

Proof. Invertibility of L_R follows from the contraction mapping principle and the following observation: let $L_1 = \sum_{|\alpha|\leq 2} c_\alpha |x|^{|\alpha|-2}\partial^\alpha$. Then

$$\|L_1\|_{\mathcal{H}^2_a \to \mathcal{H}^0_a} \leq C(n,a) \sup_{|\alpha|\leq 2, x\in\mathcal{K}} |c_\alpha(x)|.$$

In a standard fashion, we write

$$L_R = L_0\left(I + L_0^{-1}(L_R - L_0)\right),$$

where I is the identity mapping from $\overset{\circ}{\mathcal{H}}{}^2_a$ to $\overset{\circ}{\mathcal{H}}{}^2_a$. It is clear that $\|(L_0)^{-1}(L_R - L_0)\| < 1$ for R large enough. □

This proposition together with the estimate of Lemma 3.3.6 gives the existence result where the integral norms are replaced by the pointwise estimates.

Proposition 3.3.11. *Let $|f(x)| \leq K|x|^b$ for $x \in \mathcal{K}_{R,\infty}$ and $f(x) = 0$ for $x \in \mathcal{K}_{0,R}$. Let R be large enough. Then there exists $u \in \bigcap_{a<-n-2b} \overset{\circ}{\mathcal{H}}{}^2_a$ such that*

$$L_R u = f. \tag{3.3.18}$$

Moreover,

$$|u(x)| = O(|x|^{b+2+\gamma}) \quad as \quad x \to \infty$$

for all $\gamma > 0$.

Now we are ready to justify the existence of the functions $\mathcal{P}[v]$ introduced in the beginning.

Proposition 3.3.12. *For any* $\lambda \in T_L$ *there exists* $R = R(\lambda)$ *such that for any* $v \in Y(\lambda)$ *there exists a solution* $\mathcal{P}[v]$ *to* $(L_R\mathcal{P}[v], \mathcal{P}[v]|_{\partial\mathcal{K}}) = (0,0)$ *satisfying*

$$\mathcal{P}[v] - v = O(|x|^{-\operatorname{Im}\lambda-\gamma}) \quad as \quad x \to \infty$$

for all $\gamma < \alpha$.

Proof. We look for $\mathcal{P}[v]$ in the form $\mathcal{P}[v] = v + z$. The resulting equation for z reads as

$$L_R z = (L_0 - L_R)v := f.$$

The right-hand side of the last expression satisfies $|f(x)| \le C|x|^{-2-\operatorname{Im}\lambda+\varepsilon}$ for any $\varepsilon > 0$, and $f(x) \equiv 0$ if $|x| < R$. We apply Proposition 3.3.11 to finish the proof. □

Lemma 3.3.13. *Let* $v \in \mathring{\mathcal{H}}^2_a$ *satisfy the equation* $Lv = \eta$, *where* $\operatorname{supp}\eta \in \mathcal{K}_{R,2R}$, $R > 0$ *and* $\eta \in L^2(\mathcal{K}_{R,2R})$. *Let* $v = 0$ *outside* $\mathcal{K}_{R,\infty}$. *Take* $b > a$ *such that the line* $\operatorname{Im}\lambda = \frac{b+n-4}{2}$ *does not cross* T_L. *Then*

$$v = \theta_R \sum_{\lambda} \mathcal{P}[f_\lambda] + O(r^{\frac{4-n-b}{2}}), \quad f_\lambda \in Y(\lambda),$$

where the summation is over all $\lambda \in T_L$ *lying in the strip* $\frac{a+n-4}{2} \le \operatorname{Im}\lambda < \frac{b+n-4}{2}$.

Proof. We assume that the line $\operatorname{Im}\lambda = \frac{a+n-4}{2}$ does not cross T_L. Otherwise, since v is zero in the neighbourhood of 0, $v \in \mathring{\mathcal{H}}^2_{a'}$ for any $a' < a$. Then the argument below can be carried out starting from $a' = a - \varepsilon$ with sufficiently small $\varepsilon > 0$. Let us choose the sequence $\gamma_j \in (0, \alpha]$ such that $\sum_{j=1}^{\infty} \gamma_j = +\infty$ and all lines $\operatorname{Im}\lambda = \frac{a+n-4}{2} + \sum_{j=1}^{k} \gamma_j$ do not cross T_L. Denote

$$a_0 = a, \; a_k = a + \sum_{j=1}^{k} \gamma_j, \; h_k = \frac{a_j + n - 4}{2}.$$

We "freeze" the coefficients to obtain

$$L_0 v = (L_0 - L)v + \eta := F_1,$$

where the right-hand side $F_1 \in \mathcal{H}^0_{a+2\gamma}$ for all $\gamma \le \alpha$. Choosing $\gamma = \gamma_1$ and applying Lemma 3.3.5, we obtain

$$v = \sum_\lambda f_\lambda + v_{1,0} := \mathcal{S}_1 + v_{1,0},$$

where $v_{1,0} \in \overset{\circ}{\mathcal{H}}{}^2_{a_1}$, $f_\lambda \in Y(\lambda)$, and the summation is over $\lambda \in T_L$ lying in the strip $h_0 < \operatorname{Im}\lambda < h_1$. Consider now

$$v_{1,1} = v - \theta_R \mathcal{P}[\mathcal{S}_1].$$

This function satisfies

$$Lv_{1,1} = \eta_1,$$

and $v_{1,1} \in \overset{\circ}{\mathcal{H}}{}^2_{a_1}$, $v_{1,1} = 0$ for $|x| < R$, η_1 is a function supported in $\mathcal{K}_{R,2R}$. Now we are again in the situation described in the statement of the lemma but with a replaced by a greater number a_1. Repeating this argument we construct inductively the sequence of functions $v_{k,0}, v_{k,1}$, $k = 1, 2, \dots$, such that $v_{k,0}, v_{k,1} \in \overset{\circ}{\mathcal{H}}{}^2_{a_k}$, $v_{k,1} = 0$ outside $\mathcal{K}_{R,\infty}$,

$$v_{k-1,1} = \sum_\lambda f_\lambda + v_{k,0} := \mathcal{S}_k + v_{k,0},$$

where $f_\lambda \in Y(\lambda)$, and the sum is taken over $\lambda \in T_L \cap \{h_{k-1} < \operatorname{Im}\lambda < h_k\}$,

$$v_{k,1} = v_{k-1,1} - \theta_R \mathcal{P}[\mathcal{S}_k].$$

It is obvious that $v_{k,1}$ satisfy

$$Lv_{k,1} = \eta_k,$$

where η_k are square–integrable functions with support in $\mathcal{K}_{R,2R}$. Lemma 3.3.6 immediately gives the estimate:

$$v_{k,1} = O(|x|^{\frac{4-a_k-n}{2}}) \quad \text{as} \quad x \to \infty.$$

Expressing successively $v_{k-1,1}$ via $v_{k,1}$ we arrive at

$$v = \theta_R \sum_\lambda \mathcal{P}[f_\lambda] + v_{k,1}, \quad f_\lambda \in Y(\lambda),$$

where the summation is over $\lambda \in T_L \cap \{h_0 \le \operatorname{Im}\lambda < h_k\}$. It is clear that we can choose the sequence $\{\gamma_j\}$ such that on some step $a_k = b$. $\square$

3. Proofs of main results.

In the proofs of Theorems 3.3.1 and 3.3.2 we assume R to be so large, that all elliptic estimates we need hold in $\mathcal{K}_{R,\infty}$. It is so if in $\mathcal{K}_{R,\infty}$ the deviation of the coefficients of L from the coefficients of L_0 is bounded from above by a small positive constant.

Proof of Theorem 3.3.1. The function $v = u\theta_R$ satisfies the equation

$$Lv = G_1(x, v) + \eta,$$

where η is a function with support in $\mathcal{K}_{R,2R}$, $G_1(x, v) = \theta_R^\sigma G(x, v(\theta_R)^{-1})$ if $\theta_R(x) \neq 0$, and $G_1(x, v) = 0$ if $\theta_R(x) = 0$. It is easy to see that G_1 satisfies the same estimate as G: $G_1(x, v) \leq C|x|^p|v|^\sigma$. Hence, $G_1(x, v) = O(|x|^{p+\beta\sigma})$ and $G_1(x, v) = 0$ for $|x| < R$. By Proposition 3.3.9, $v \in \overset{\circ}{\mathcal{H}}{}^2_{a_1}$ for all

$$a_1 < a_1(\beta) := \min(4 - n - 2\beta, 4 - n - 2(p + \beta\sigma + 2)) = 4 - n - 2\beta.$$

"Freezing" the coefficients in the neighbourhood of infinity we obtain

$$L_0 v = (L_0 - L)v + G_1(x, v) + \eta := F.$$

From (3.3.3) it follows that $F \in \mathcal{H}^0_{a_2}$ for any

$$\begin{aligned} a_2 < a_2(\beta) &:= \min(4 - n - 2(p + \beta\sigma + 2), 4 - n - 2\beta + 2\alpha) \\ &= a_1(\beta) + 2\min(\alpha, p + 2 + \beta(1 - \sigma)). \end{aligned}$$

We choose a_1, a_2 such that $a_1 < a_1(\beta) < a_2 < a_2(\beta)$ and the lines $\operatorname{Im}\lambda = h_j = \frac{a_j+n-4}{2}$, $j = 1, 2$, do not cross T_L. We also assume that either there are no numbers from T_L in the strip $\Pi = \{\lambda \in \mathbb{C} : h_1 < \operatorname{Im}\lambda < h_2\}$ or all these numbers have the same imaginary part. Now we apply Lemma 3.3.5 which gives the decomposition

$$v = \sum_{\lambda \in \Pi \cap T_L} f_\lambda + v_1, \tag{3.3.19}$$

where $f_\lambda \in Y(\lambda)$ and $v_1 \in \overset{\circ}{\mathcal{H}}{}^2_{a_2}$.

Suppose that $\sum_{\lambda \in \Pi \cap T_L} f_\lambda = 0$. For example, this is so if $\Pi \cap T_L = \emptyset$. Then $v = v_1$, whence $v \in \overset{\circ}{\mathcal{H}}{}^2_{a_2}$. Lemma 3.3.6 yields the estimate $v = O(|x|^{\beta - \delta_1})$, where $\delta_1 = \frac{a_2 - a_1(\beta)}{2} > 0$. We see that we are in the situation described in the statement of the theorem but with lesser β. In this case we repeat the argument.

Let $\sum_{\lambda \in \Pi \cap T_L} f_\lambda \neq 0$. Choose $\lambda \in \Pi \cap T_L$ and denote $h = \operatorname{Im}\lambda$. Note that v_1 satisfies

$$Lv_1 = G_1(x, v) + \eta + (L_0 - L)f_{\lambda_0}.$$

Hence, $Lv_1 = O(|x|^\gamma)$ for any $\gamma > \max(-h - 2 - \alpha, p + \beta\sigma)$. Lemma 3.3.6 yields the estimate $v_1 = O(|x|^{-h-\delta_2})$, where $\delta_2 = \frac{a_2+n-4}{2} - h > 0$. Therefore, $v = O(|x|^{-h+\varepsilon})$ for any $\varepsilon > 0$. This, in turn, implies that $Lv = O(|x|^{p-\sigma h+\varepsilon})$ for all $\varepsilon > 0$.

Let a_3 be any number lesser than $-n-2(p-\sigma h)$ such that the line $\operatorname{Im}\lambda = \frac{a_3+n-4}{2}$ does not cross T_L. We have $Lv \in \overset{\circ}{\mathcal{H}}{}^2_{a_3}$ since $Lv = 0$ for $|x| < R$. Applying Lemma 3.3.11, we can find a solution to $Lw = Lv$ such that $w \in \overset{\circ}{\mathcal{H}}{}^2_{a_3}$ and

$$w = O(|x|^{\frac{a_3+n-4}{2}}) = O(|x|^{2+p-\sigma h+\varepsilon})$$

where $\varepsilon > 0$ can be made arbitrarily small by choosing appropriate a_3. Consider now $v_2 = v - w\theta_R$. Choose $a_4 < 2h+4-n$ such that there are no numbers from T_L in the strip $\frac{a_4+n-4}{2} < \operatorname{Im}\lambda < h$. It is clear that $v_2 \in \overset{\circ}{\mathcal{H}}{}^2_{a_4}$. For v_2 we get the equation

$$Lv_2 = \eta_2,$$

where $supp\, \eta_2 \subset \mathcal{K}_{R,2R}$, and $\eta_2 \in \mathcal{H}^0_s$ for any s. We finish the proof be applying Lemma 3.3.13 to v_2 with b such that $\frac{4-n-b}{2} = 2+p-\sigma h+\varepsilon$, where $\varepsilon > 0$ can be arbitrarily small. The resulting decomposition $v = v_2 + \theta_R w$ is the required one. $\square$

Proof of Theorem 3.3.2. Let us choose some $R > 1$, without fixing it (we will do it later). We choose also the positive number

$$\gamma < -\operatorname{Im}\lambda_0(1-\sigma) - 2 - p,$$

such that

$$-\operatorname{Im}\lambda > -\operatorname{Im}\lambda_0 - \gamma$$

for all $\lambda \in \Pi(\lambda_0) \cap T_L$. We denote

$$\varepsilon = -\operatorname{Im}\lambda_0(1-\sigma) - 2 - p - \gamma,$$

$$Q = \sum_{\lambda\in\Pi(\lambda_0)\cap T_L} \mathcal{P}[f_\lambda],$$

$$C_0 = \sup_{x\in\mathcal{K}_{R,\infty}} |Q(x)| \cdot |x|^{\operatorname{Im}\lambda_0} (\ln|x|)^{1-m(\lambda_0)}.$$

Let us introduce the Banach space $\mathfrak{B}$ which consists of all $v \in C(\overline{\mathcal{K}})$ such that 1) $v(x)$ is zero on $\partial\mathcal{K}$, 2) $v(x) = 0$ if $|x| < R$, and 3) $\|v\|_{\mathfrak{B}} := \sup_{x\in\mathcal{K}} |v(x)| \cdot |x|^{\operatorname{Im}\lambda_0+\gamma} < \infty$. We denote $\mathfrak{S}_1 = \{v \in \mathfrak{B} : \|v\|_{\mathfrak{B}} \le 1\}$. For $v \in \mathfrak{B}$ we denote

$$F(v) = G(x, \theta_R Q + v).$$

It is clear that $F(v) \equiv 0$ when $|x| < R$ and

$$|F(v)| \le C\left(C_0^\sigma + \|v\|_{\mathfrak{B}}^\sigma |x|^{-\gamma\sigma}\right)|x|^{-\sigma\operatorname{Im}\lambda_0+p}(\ln|x|)^{\sigma(m(\lambda_0)-1)}.$$

Hence, for any $\delta > 0$ and $a = -n - 2i\lambda_0\sigma - 2p - \delta$,

$$\|F(v)\|^2_{\overset{\circ}{\mathcal{H}}{}^0_a} \le C\left(\|v\|_{\mathfrak{B}}^{2\sigma} R^{-\delta-2\gamma\sigma} + C_0^{2\sigma} R^{-\delta}\right)(\ln R)^{2\sigma(m(\lambda_0)-1)}$$
$$:= (\psi(\|v\|_{\mathfrak{B}}, R))^2.$$

Further, for $v \in \mathfrak{B}$ we denote

$$Tv = \theta_R \left(L_R\right)^{-1} F(v),$$

where L_R is considered as an operator from $\overset{\circ}{\mathcal{H}}{}^2_a$ to $\mathcal{H}^0_a$ for the given choice of a. Lemma 3.3.6 yields that

$$\begin{aligned}|Tv(x)| \leq C \Big[\psi(\|v\|_{\mathfrak{B}}, R)|x|^{2-\operatorname{Im}\lambda_0\sigma+p+\delta/2} \\ + \left(C_0^\sigma + \|v\|^\sigma_{\mathfrak{B}}|x|^{-\gamma\sigma}\right)|x|^{2-\operatorname{Im}\lambda_0\sigma+p}(\ln|x|)^{\sigma(m(\lambda_0)-1)}\Big] \\ \leq C\psi(\|v\|_{\mathfrak{B}}, R)|x|^{2-\operatorname{Im}\lambda_0\sigma+p+\delta/2}.\end{aligned}$$

Hence, if R is large enough, and δ is sufficiently small, then $T\mathfrak{S}_1 \subset \mathfrak{S}_1$ and

$$|Tv(x)| \leq |x|^{-\operatorname{Im}\lambda_0-\gamma-\tau}, \tag{3.3.20}$$

where

$$\tau = (-\operatorname{Im}\lambda_0)(1-\sigma) - 2 - p - \delta/2 - \gamma = \varepsilon - \delta/2 > 0.$$

By (3.3.11), for $v_1, v_2 \in \mathfrak{S}_1$ we have

$$|F(v_1) - F(v_2)| \leq C(C_0 \left(\ln R\right)^{m(\lambda_0)-1} + R^{-\gamma})^{\sigma-1}|x|^{-\operatorname{Im}\lambda_0\sigma+p-\gamma}\|v_1 - v_2\|_{\mathfrak{B}}.$$

This estimate together with Lemma 3.3.6 yields

$$\begin{aligned}&|Tv_1(x) - Tv_2(x)| \\ &\leq C(C_0 \left(\ln R\right)^{m(\lambda_0)-1} + R^{-\gamma})^{\sigma-1}\|v_1 - v_2\|_{\mathfrak{B}} R^{-\gamma-\delta/2}|x|^{2+p-\operatorname{Im}\lambda_0\sigma+\delta/2} \\ &\leq C(C_0 \left(\ln R\right)^{m(\lambda_0)-1} + \rho R^{-\gamma})^{\sigma-1}\|v_1 - v_2\|_{\mathfrak{B}} R^{-\gamma-\varepsilon}|x|^{-\operatorname{Im}\lambda_0-\gamma}.\end{aligned}$$

Hence, T is a continuous operator from $\mathfrak{S}_1$ to $\mathfrak{S}_1$. Next, we show the precompactness of $T\mathfrak{S}_1$. We demonstrate that for any $\varepsilon > 0$ we can construct the finite ε – net for $T\mathfrak{S}_1$ in $\mathfrak{B}$.

By (3.3.20), for any $\varepsilon > 0$ we can find $R_1 > R$ such that for all $v \in \mathfrak{S}_1$

$$\|v - v\chi_1\|_{\mathfrak{B}} < \varepsilon/2.$$

Here χ_1 stands for the characteristic function of the ball of radius R_1 centered at 0.

Denote $T_1 v = (L_R)^{-1}F(v)$. For any $v \in \mathfrak{S}_1$

$$\sup_{x\in\mathcal{K}_{R,R_1}} |T_1 v(x)| \leq C_1,$$

$$\sup_{x\in\mathcal{K}_{R,R_1}} |L_R T_1 v(x)| \leq C_2.$$

with C_1, C_2 independent of v. By construction, $T_1 v \in W^{2,2}_{loc}(\overline{\mathcal{K}}/\{0\})$. From Lemma 3.3.7 it follows that for all $v \in \mathfrak{S}_1$

$$\|T_1 v\|_{W^{2,2n}(\mathcal{K}_{R,R_1})} \leq C_3,$$

where C_3 is independent of v. It is clear that

$$\|Tv\|_{W^{2,2n}(\mathcal{K}_{R,R_1}))} \leq C\|T_1 v\|_{W^{2,2n}(\mathcal{K}_{R,R_1})}.$$

According to the Sobolev imbedding theorem,

$$\|\cdot\|_{C^1(\mathcal{K}_{R,R_1})} \leq \|\cdot\|_{W^{2,2n}(\mathcal{K}_{R,R_1})}.$$

Hence, for all $v \in \mathfrak{S}_1$

$$\|Tv\|_{C^1(\mathcal{K}_{R,R_1})} \leq C_4,$$

where C_4 is independent of v. The precompactness of $T\mathfrak{S}_1$ in $C(\mathcal{K}_{R,R_1})$ now follows from the Arzela-Ascoli theorem.

Note that if $u \in \mathfrak{B}$ and $u(x) = 0$ outside $\mathcal{K}_{R,R_1}$ then

$$\|u\|_{\mathfrak{B}} \leq C_5\|u\|_{C(\mathcal{K}_{R,R_1})},$$

where $C_5 = R^{\operatorname{Im}\lambda_0+\gamma}$ if $\operatorname{Im}\lambda_0 + \gamma < 0$ and $C_5 = R_1^{\operatorname{Im}\lambda_0+\gamma}$ if $\operatorname{Im}\lambda_0 + \gamma \geq 0$.

Let $\{f_i\}_{i=1}^{N(\varepsilon)}$ be a $\frac{\varepsilon}{2C_5}$ – net for $T\mathfrak{S}_1$ in $C(\mathcal{K}_{R,R_1})$. Then $\{g_i\}_{i=1}^{N(\varepsilon)}$, where $g_i(x) = f_i(x)$, if $x \in \mathcal{K}_{R,R_1}$ and $g_i(x) = 0$, if $x \notin \mathcal{K}_{R,R_1}$, is an ε – net for $T\mathfrak{S}_1$ in $\mathfrak{B}$.

Now we have made all the preparations to apply the Leray-Schauder principle which implies the existence of $v_f \in \mathfrak{S}_1$ such that $Tv_f = v_f$. The sum $u = v_f + Q$ solves the equation (3.3.1) in $\mathcal{K}_{2R,\infty}$. From the standard elliptic regularity theory it follows that $u \in W^{2,n}_{loc}(\overline{\mathcal{K}}_{2R,\infty})$ and u satisfies the boundary condition (3.3.2) in the classical sense. $\square$

Chapter 4

Nonlinear Degenerate Parabolic Equations

4.1 A new proof of the Hölder continuity of solutions to p–Laplace type parabolic equations.

This section follows line by line my joint paper with U. Gianazza and V. Vespri [46].

1. Introduction and Main Result

Let Ω be a domain in $\mathbb{R}^n$. For $T > 0$ let Ω_T denote the cylindrical domain $\Omega \times (0, T]$. In the cylinder Ω_T consider the quasi–linear parabolic differential equation

$$u_t = \operatorname{div} \mathbf{A}(x, t, u, Du). \tag{4.1.1}$$

The function $\mathbf{A} : \Omega_T \times \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a Carathéodory function satisfying

$$\mathbf{A}(x, t, u, Du) \cdot Du \geq C_o |Du|^p, \tag{4.1.2}$$

$$|\mathbf{A}(x, t, u, Du)| \leq C_1 |Du|^{p-1} \tag{4.1.3}$$

almost everywhere in Ω_T for $p > 2$ and where C_o, C_1 are given positive constants.

A function

$$u \in C_{loc}(0, T; L^2_{loc}(\Omega)) \cap L^p_{loc}(0, T; W^{1,p}_{loc}(\Omega))$$

is a local weak super(sub)solution to (4.1.1) if for every compact set $K \subset \Omega$ and for every subinterval $[t_1, t_2] \subset (0, T]$ one has

$$\int_K u\varphi dx \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K [-u\varphi_t + \mathbf{A}(x, t, u, Du) \cdot D\varphi] \, dxdt \geq (\leq) 0$$

for all nonnegative test functions

$$\varphi \in W^{1,2}_{loc}(0,T;L^2(K)) \cap L^p_{loc}(0,T;W^{1,p}_o(K)).$$

We say that u is a local weak solution if it is both a local weak sub- and supersolution.

We say that a constant $\gamma = \gamma(data)$ if it can be quantitatively expressed in terms of n, p, C_o, C_1. For $y \in \mathbb{R}^n$ and $\rho > 0$ let

$$K^y_\rho = \{x \in \mathbb{R}^n : |x_i - y_i| < \rho/2,\ i = 1, \dots, n\}.$$

For a given cylinder $Q = K^x_\rho \times [t_o - \theta\rho^p, t_o]$ we denote $\frac{1}{2}Q = K^x_{\rho/2} \times [t_o - \theta(\rho/2)^p, t_o]$. The main result of this paper is the following

Theorem 4.1.1. *Let u be a locally bounded weak solution of* (4.1.1) *in Ω_T. Then, up to modification on a set of measure zero, u is locally Hölder continuous in $\Omega \times (0,T]$. The Hölder constants can be determined a priori only in terms of the data.*

Indeed, that locally bounded weak solutions to (4.1.1) are locally Hölder continuous is not a new result: the proof of this fact was first given by E. DiBenedetto in [23] for the degenerate case $p > 2$, and by Y.Z. Chen and E. DiBenedetto for the singular case $1 < p < 2$ in [11], [12]. The book [25] contains the proof of the Hölder continuity of solutions for equations with a very general structure. The main ideas underlying the original proof by DiBenedetto, namely the so–called *intrisic scaling method*, are discussed in [31]. A thorough presentation of this same set of techniques is given in the recent monograph [109].

Here the focus is on the degenerate case, i.e. when $p > 2$; the corresponding approach to the Hölder continuity for the singular case, namely when $1 < p < 2$, will be dealt with in [30].

The structure of the proof given in [23] is based on studying separately two cases. Either one can find a cylinder of the type $K^{x_o}_\rho \times [t_o - \theta\rho^p, t_o]$ where u is mostly large, or such a cylinder cannot be found. In either case the conclusion is that the essential oscillation of u in a smaller cylinder about (x_o, t_o) decreases in a way that can be quantitatively measured.

The actual technical implementation of the previous alternative is not an easy job; the point in giving a new proof of the by–now classical result by DiBenedetto is to show how a certain set of ideas, which led to the proof of the Harnack inequality in [28], can simplify the argument, and avoid any use of alternatives. We believe that the new proof has a further significant feature, namely its strong geometric character.

Three final comments are due here:

- In order to present the essence of the approach, only the case of homogeneous–structure equations is dealt with here, but with little further effort the full quasi–linear case could be considered too.
- In the following we deal with equations of p–laplacian type, but, with some care, the same kind of arguments can be used to prove the Hölder continuity of quasi–linear parabolic differential equation of the sort

  $$u_t = \operatorname{div} \mathbf{A}(x, t, u, Du),$$

  where the function $\mathbf{A} : \Omega_T \times \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a Carathéodory function satisfying the structure conditions

  $$\mathbf{A}(x, t, u, Du) \cdot Du \geq C_o |u|^{m-1} |Du|^2 \tag{4.1.4}$$
  $$|\mathbf{A}(x, t, u, Du)| \leq C_1 |u|^{m-1} |Du| \tag{4.1.5}$$

  almost everywhere in Ω_T for $m > 1$, C_o and C_1 being given positive constants. The prototype of this kind of equations is the so–called *porous medium equation*, namely

  $$u_t - \Delta |u|^{m-1} u = 0,$$

  which has been extensively studied in the last thirty years, in the context of non–linear diffusion phenomena. For a thorough treatment of this very interesting topic, see for example [112]. The care we were referring above, is due to the fact that if u is a solution to a porous medium equation, given a generic constant $c \neq 0$, in general $u + c$ is not a solution. Therefore one has to take into account that solutions are signed solutions, and this brings about some further technical difficulties.
- In the rest of the paper by solutions we will always mean weak solutions.

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2. Main Lemma and Proof of Theorem 4.1.1

As it will be clear at the end of this Section, the proof of Theorem 4.1.1 is a straightforward consequence of the following lemma.

Lemma 4.1.2. *Let u be a nonnegative solution to* (4.1.1) *in the cylinder $Q = K_4^0 \times (-2, T]$. There exist constants $0 < \gamma_1 < \gamma_2$ and $\mu > 0$, depending only on C_o, C_1, n, p, such that if $T \geq \gamma_2$ and*

$$|\{(x, t) \in K_1^0 \times (-1, 0] : u(x, t) \geq \frac{1}{2}\}| \geq \frac{1}{2}$$

then

$$\operatorname*{ess\,inf}_{Q'} u \geq \mu, \quad \textit{where} \quad Q' = K_1^0 \times (\gamma_1, \gamma_2].$$

The proof of this lemma is the most technically involved part of the paper and we postpone its proof to the last Section of the paper. Let

$$\gamma_3 = \gamma_2 - \gamma_1, \quad \gamma_4 = 2 + \gamma_2.$$

It is obvious that $0 < \gamma_3 < \gamma_4$. The following proposition is elementary and we skip its proof.

Proposition 4.1.3. *Let u be a solution to (4.1.1) in the cylinder $Q = K_{4\rho}^{x_o} \times [t_o - \gamma_4 \omega^{2-p} \rho^p, t_o]$, where $\omega > 0$. Let $\beta \in \mathbb{R}$. Set*

$$z = \pm \frac{u + \beta}{\omega}, \quad x = x_o + \rho y, \quad t = t_o + (\tau - \gamma_2)\omega^{2-p}\rho^p;$$

then z is a solution to the equation

$$z_\tau = \operatorname{div} \mathbf{A}_1(y, \tau, z, Dz)$$

in the cylinder $Q_1 = K_4^0 \times [-2, \gamma_2]$, where $\mathbf{A}_1$ is a Carathéodory function satisfying

$$\mathbf{A}_1(y, \tau, z, Dz) \cdot Dz \geq C_o |Dz|^p,$$
$$|\mathbf{A}_1(y, \tau, z, Dz)| \leq C_1 |Dz|^{p-1}$$

The following corollary of Lemma 4.1.2 is a classical step in the proof of the Hölder continuity of solutions to degenerate and singular parabolic partial differential equations.

Lemma 4.1.4. *Let u be a solution to (4.1.1) in the cylinder $Q = K_{4\rho}^{x_o} \times [t_o - \gamma_4 \omega^{2-p} \rho^p, t_o]$, where $\omega > 0$. Let $\operatorname{ess\,osc}_Q u \geq \omega$. Let $Q' = K_\rho^{x_o} \times [t_o - \gamma_3 \omega^{2-p} \rho^p, t_o]$. Then*

$$\operatorname*{ess\,osc}_{Q'} u \leq \operatorname*{ess\,osc}_{Q} u - \mu\omega,$$

where μ is the quantity given by Lemma 4.1.2

Proof. Let

$$m = \operatorname*{ess\,inf}_Q u, \quad M = \operatorname*{ess\,sup}_Q u, \quad \xi = \frac{M - m}{\omega} \geq 1,$$

and

$$z = \frac{u - m}{\omega}, \quad x = x_o + \rho y, \quad t = t_o + (\tau - \gamma_2)\omega^{2-p}\rho^p.$$

By Proposition 4.1.3, z is a non–negative solution to

$$z_\tau = \operatorname{div} \mathbf{A}_1(y, \tau, z, Dz)$$

in the cylinder $Q_1 = K_4^0 \times [-2, \gamma_2]$, and the vector field $\mathbf{A}_1$ satisfies the same structural conditions as $\mathbf{A}$. One of the following statements holds:

$$\left|\left\{z \geq \frac{\xi}{2}\right\} \cap K_1^0 \times [-1, 0]\right| \geq \frac{1}{2}, \tag{4.1.6}$$

$$\left|\left\{z \geq \frac{\xi}{2}\right\} \cap K_1^0 \times [-1, 0]\right| < \frac{1}{2}. \tag{4.1.7}$$

Let (4.1.6) hold: then $\left|\left\{z \geq \frac{1}{2}\right\} \cap K_1^0 \times [-1, 0]\right| \geq \frac{1}{2}$, and by Lemma 4.1.2 we obtain

$$z \geq \mu \quad \text{a.e. in} \quad K_1^0 \times]\gamma_1, \gamma_2].$$

Hence

$$\operatorname*{ess\,inf}_{Q'} u \geq m + \mu\omega.$$

and the assertion of the lemma easily follows.
Analogously, if (4.1.7) holds, then the function

$$\hat{z} = \xi - z = \frac{M - u}{\omega}$$

satisfies

$$\hat{z} \geq \mu \quad \text{a.e. in} \quad K_1^0 \times [\gamma_1, \gamma_2],$$

and

$$\operatorname*{ess\,sup}_{Q'} u \leq M - \mu\omega. \qquad \square$$

The proof of the following energy inequalities can be found in [25].

Proposition 4.1.5. *Let the cylinder $Q = K_\rho^y \times [t_1, t_2] \subset \Omega_T$ and ξ be a non–negative piecewise–smooth test function vanishing on the lateral boundary of Q. If u is a subsolution to the equation* (4.1.1) *in Ω_T, then for any $k \in \mathbb{R}$ we have*

$$\int_{K_\rho^y} (u-k)_+^2 \xi^p dx \Big|_{t_1}^{t_2} + C_o \iint_Q |D(u-k)_+|^p \xi^p dxdt$$
$$\leq p \iint_Q (u-k)_+^2 \xi^{p-1} \xi_t dxdt + \tilde{\gamma}_o \iint_Q (u-k)_+^p |D\xi|^p dxdt. \tag{4.1.8}$$

Proposition 4.1.6. *Let the cylinder $Q = K_\rho^y \times [t_1, t_2] \subset \Omega_T$ and ξ be a non–negative piecewise–smooth test function vanishing on the lateral boundary of Q.*

If u is a supersolution to (4.1.1) in Ω_T, then for any $k \in \mathbb{R}$ we have

$$\int_{K_\rho^y}(u-k)_-^2\xi^p dx\Big|_{t_1}^{t_2} + C_o\iint_Q |D(u-k)_-|^p\xi^p dxdt$$
$$\le p\iint_Q (u-k)_-^2\xi^{p-1}\xi_t dxdt + \tilde{\gamma}_o\iint_Q (u-k)_-^p|D\xi|^p dxdt. \qquad (4.1.9)$$

Remark 4.1.7. *The proof shows that in both cases*

$$\tilde{\gamma}_o = (2C_1)^p\left(\frac{p-1}{C_o}\right)^{p-1}.$$

Assuming for the moment the validity of Lemma 4.1.2, we proceed with the

Proof of Theorem 4.1.1. Let $(x_o,t_o)\in\Omega_T$ and set

$$d_x = dist(x_o,\partial\Omega), \quad d_t = t_o > 0.$$

Assume that $\sup_{\Omega_T}|u| = M < \infty$. Let $\{\omega_j\}_{j=0}^\infty$, $\{\rho_j\}_{j=0}^\infty$ be the sequences of positive numbers defined by

$$\rho_j = \varepsilon\rho_{j-1}, \quad \omega_j = \delta\omega_{j-1},$$

where $\delta\in(0,1)$, $\varepsilon\in(0,\frac{1}{4}]$, $\varepsilon<\delta$ are to be chosen. Notice that the condition $\varepsilon<\delta$ guarantees that the sequence of cylinders shrinks to a point. We also require that $\rho_o\le d_x$ and $\gamma_4\omega_o^{2-p}\left(\frac{\rho_o}{4}\right)^p\le d_t$. Let

$$Q_j = K_{\rho_j}^{x_o}\times\left[t_o-\gamma_4\omega_j^{2-p}\left(\frac{\rho_j}{4}\right)^p, t_o\right],$$

and denote

$$A_j = \operatorname*{ess\,osc}_{Q_j} u.$$

We want to show that there exists a constant $\Lambda=\Lambda(data)>1$, such that

$$A_j\le\Lambda\omega_j \qquad \Rightarrow \qquad A_{j+1}\le\Lambda\omega_{j+1}.$$

Suppose that ε and δ are such that $\gamma_4\varepsilon^p\delta^{2-p}\le\gamma_3$.
Assume first that $A_j\ge\omega_j$. Then, in virtue of Lemma 4.1.4,

$$A_{j+1}\le A_j-\mu\omega_j\le(\Lambda-\mu)\omega_j = \frac{\Lambda-\mu}{\delta\Lambda}\Lambda\omega_{j+1}$$
$$\le\Lambda\omega_{j+1} \quad \text{if} \quad \Lambda-\mu\le\delta\Lambda\Leftrightarrow\Lambda\le\frac{\mu}{1-\delta}.$$

On the other hand, if $A_j\le\omega_j$, then

$$A_{j+1}\le A_j\le\omega_j = \frac{1}{\delta}\omega_{j+1}\le\Lambda\omega_{j+1} \quad \text{if} \quad \frac{1}{\delta}\le\Lambda.$$

Hence, any Λ such that
$$\frac{1}{\delta} \le \Lambda \le \frac{\mu}{1-\delta}.$$
will do. It is clear that the previous inequality is satisfied if
$$\frac{1}{\delta} \le \frac{\mu}{1-\delta} \qquad \Leftrightarrow \qquad \delta \ge \frac{1}{1+\mu}.$$
Take $\delta = \frac{1}{1+\mu}$ and $\Lambda = 1+\mu$. Set
$$\varepsilon = \min\left\{\frac{1}{4}, \left(\frac{\gamma_3}{\gamma_4}\delta^{p-2}\right)^{\frac{1}{p}}\right\}, \quad \omega_o = 2M, \quad \rho_o = \min\left\{d_x, 4\left(\frac{d_t}{\gamma_4 \omega_o^{2-p}}\right)^{\frac{1}{p}}\right\}.$$
Then it is immediate to see that $\operatorname*{ess\,osc}_{Q_o} u \le (1+\mu)\omega_o$, which implies
$$\operatorname*{ess\,osc}_{Q_j} u \le (1+\mu)\omega_j = (1+\mu)^{1-j}\omega_o.$$
Let
$$Q_{r,s}^{x_o,t_o} = K_r^{x_o} \times (t_o - s, t_o] \subset \Omega_T,$$
and
$$\varphi(x_o, t_o, r, s) = \operatorname*{ess\,osc}_{Q_{r,s}^{x_o,t_o}} u.$$
Choosing j in such a way that $Q_{r,s}^{x_o,t_o} \subset Q_j$, we have
$$\varphi(x_o, t_o, r, s) \le (1+\mu)^2 \omega_o \max\left\{\left(\frac{s}{\gamma_4(\rho_o/4)^p \omega_o^{2-p}}\right)^{\alpha_1}, \left(\frac{r}{\rho_o}\right)^{\alpha_2}\right\}, \tag{4.1.10}$$
with
$$\alpha_1 = \frac{1}{\log_{1+\mu} \frac{1}{\varepsilon^p \delta^{2-p}}}, \quad \alpha_2 = \frac{1}{\log_{1+\mu} \frac{1}{\varepsilon}}.$$
Since $\varepsilon^p \delta^{2-p} \le \frac{\gamma_3}{\gamma_4} < 1$, and by possibly reducing μ and enlarging γ_1, we have that both α_1, $\alpha_2 \in (0,1)$. Notice that
$$\frac{\alpha_1}{\alpha_2} = \frac{1}{\log_\varepsilon \varepsilon^p \delta^{2-p}} = \frac{1}{p + (2-p)\log_\varepsilon \delta} > \frac{1}{p}.$$
The rest of the proof follows in a standard way. □

3. Auxiliary Propositions and Technical Results

In the following we gather various technical results, which are used in the proof of Lemma 4.1.2. Some of the statements will be given without proofs (and in such a case we refer the reader to [28] or [25]); others will be explicitly proved, even if the arguments are mainly proper modifications of analogous results given in [28].

The first two lemmata, which we state for sub- and supersolutions separately, are one of the traditional and most widely used tools in the regularity theory.

Lemma 4.1.8. *Let u be a subsolution to (4.1.1) in the cylinder $Q = K^y_{2\rho} \times [t_1 - \theta(2\rho)^p, t_1]$. Let $\mu_+ \geq \operatorname{ess\,sup}_Q u(x,t)$. Then for any $\omega > 0$ and $a \in (0,1)$ there exists a number s, which depends only on the data, a, and $\theta\omega^{p-2}$, such that if*

$$|\{(x,t) \in Q \,:\, u(x,t) > \mu_+ - \omega\}| \leq s|Q|,$$

then we have

$$\operatorname*{ess\,sup}_{\frac{1}{2}Q} u(x,t) \leq \mu_+ - a\omega.$$

Lemma 4.1.9. *Let u be a supersolution to (4.1.1) in the cylinder $Q = K^y_{2\rho} \times [t_1 - \theta(2\rho)^p, t_1]$. Let $\mu_- \leq \operatorname{ess\,inf}_Q u(x,t)$. Then for any $\omega > 0$ and $a \in (0,1)$ there exist a number s, which depends only on the data, a, and $\theta\omega^{p-2}$, such that if*

$$|\{(x,t) \in Q \,:\, u(x,t) < \mu_- + \omega\}| \leq s|Q|$$

then we have

$$\operatorname*{ess\,inf}_{\frac{1}{2}Q} u(x,t) \geq \mu_- + a\omega.$$

The proof shows that the value of s in Lemmas 4.1.8 and 4.1.9 is the same for the same values of a and $\theta\omega^{p-2}$, namely

$$s = \left(\frac{1-a}{\gamma(data)}\right)^{n+p} \frac{[\theta\omega^{p-2}]^{n/p}}{[1+\theta\omega^{p-2}]^{(n+p)/p}}.$$

Given a and $\xi = \theta\omega^{p-2}$ we denote the corresponding value of s by $s(a,\xi)$.

The next lemma is a variant of the previous result.

Lemma 4.1.10. *Let u be a non–negative supersolution to (4.1.1) in the cylinder $Q = K^y_{2\rho} \times [t_1 - \theta(2\rho)^p, t_1]$. Suppose that $\operatorname{ess\,inf}_{K^y_{2\rho}} u(x, t_1 - \theta(2\rho)^p) \geq k$. Then there exists $\nu = \nu(data) > 0$ such that if $\theta < \nu k^{2-p}$,*

$$\operatorname*{ess\,inf}_{K^y_\rho} u(x,t_1) \geq k/2.$$

A consequence of the previous lemma is

Corollary 4.1.11. *Let u be a non–negative supersolution to (4.1.1) in the cylinder $K^y_{2\rho} \times [t_1, t_1+T]$. Let $\operatorname{ess\,inf}_{K^y_{2\rho}} u(x,t_1) \geq k$. Then for all $t \in (t_1, t_1+T]$ we have*

$$\operatorname*{ess\,inf}_{K^y_\rho} u(x,t) \geq \frac{k}{2}\left(1 + \frac{t-t_1}{\nu k^{2-p}(2\rho)^p}\right)^{\frac{1}{2-p}}, \tag{4.1.11}$$

where ν is the constant from the statement of Lemma 4.1.10.

Proof. It is clear, that for any $\tau \in [0,1]$ we have

$$\operatorname*{ess\,inf}_{K^y_{2\rho}} u(x,t_1) \geq \tau k. \tag{4.1.12}$$

If $t - t_1 \leq \nu k^{2-p}(2\rho)^p$, then Lemma 4.1.10 yields $u(x,t) \geq k/2$ a.e. in K^y_ρ. Now assume that $t - t_1 > \nu k^{2-p}(2\rho)^p$. In (4.1.12) take

$$\tau = \left(\frac{\nu k^{2-p}(2\rho)^p}{t-t_1}\right)^{\frac{1}{p-2}}$$

and apply Lemma 4.1.10 with k replaced by τk. This gives

$$u(x,t) \geq \left(\frac{\nu k^{2-p}(2\rho)^p}{t-t_1}\right)^{\frac{1}{p-2}} \frac{k}{2} \quad \text{a.e. in} \quad K^y_\rho.$$

The combination of the estimates for $t \leq t_1 + \nu k^{2-p}(2\rho)^p$ and $t > t_1 + \nu k^{2-p}(2\rho)^p$ concludes the proof. $\square$

The next lemma is analogous to Proposition 6.1 of [28].

Lemma 4.1.12. *Let v be a non-negative supersolution of* (4.1.1) *in the cylinder $Q = K^0_4 \times [0,T]$. Assume we have*

$$|\{x \in K^0_2 : v(x,t) \geq 1\}| \geq \alpha |K^0_2|,$$

for all $t \in [0,T]$, where $\alpha \in (0,1)$ is a given constant. Then for any $\varepsilon > 0$ there exist $\theta = \theta(\alpha, \varepsilon, data) > 0$ such that if $T \geq \theta 2^{p+1}$, then for the cylinder $Q_1 = K^0_2 \times [\theta 2^p, \theta 2^{p+1}] \subset Q$, we have

$$|\{(x,t) \in Q_1 : v(x,t) < \theta^{\frac{1}{2-p}}\}| \leq \varepsilon |Q_1|.$$

Moreover, θ is a monotone decreasing function of ε. Given α and ε we denote the corresponding θ by $\theta(\varepsilon, \alpha)$.

Proof. Denote $k_j = 2^{-j}$ for $j = 0, 1, \ldots, j_*$, where j_* will be chosen later, and let $Q_2 = K^0_4 \times [0, 2^{p+1}\theta]$, where the constant θ will be specified later. Take the piecewise-smooth cut-off function $\xi(x,t)$ such that $\xi = 1$ on Q_1, $0 \leq \xi \leq 1$ on Q_2, ξ vanishes on the parabolic boundary of Q_2, $|\xi_t| \leq \frac{2}{2^p \theta}$ and $|D\xi| \leq 1$.

From inequality (4.1.9), we obtain

$$\iint_{Q_1} |D(v-k_j)_-|^p dxdt \leq \gamma |Q_2| \left(\frac{k_j^2}{\theta} + k_j^p\right).$$

Take $\theta = k_{j_*}^{2-p}$. Then the last inequality yields

$$\iint_{Q_1} |D(v-k_j)_-|^p dxdt \le \gamma |Q_1| k_j^p.$$

Denote

$$A_j = \{(x,t) \in Q_1 : v(x,t) < k_j\} \quad \text{and} \quad A_j(\tau) = \{x \in K_2^0 : v(x,\tau) < k_j\}.$$

Using De Giorgi - Poincaré inequality (see [25], Chapter I, Lemma 2.2), thanks to the hypotheses we obtain

$$\begin{aligned}(k_j - k_{j+1})|A_{j+1}(\tau)| &\le \frac{\gamma(n)}{|K_2^0 \setminus A_j(\tau)|} \int_{A_j(\tau)\setminus A_{j+1}(\tau)} |D(v-k_j)_-| dx \\ &\le \frac{\gamma(n)}{\alpha |K_2^0|} \int_{A_j(\tau)\setminus A_{j+1}(\tau)} |D(v-k_j)_-| dx.\end{aligned}$$

Integration of the last inequality over $\tau \in [2^p\theta, 2^{p+1}\theta]$ yields

$$\begin{aligned}\frac{k_j}{2}|A_{j+1}| &\le \frac{\gamma}{\alpha} \iint_{A_j\setminus A_{j+1}} |D(v-k_j)_-| dxdt \\ &\le \frac{\gamma}{\alpha}\left(\iint_{A_j\setminus A_{j+1}} |D(v-k_j)_-|^p dxdt\right)^{\frac{1}{p}} |A_j \setminus A_{j+1}|^{\frac{p-1}{p}} \\ &\le \frac{\gamma}{\alpha} k_j |Q_1|^{\frac{1}{p}} |A_j \setminus A_{j+1}|^{\frac{p-1}{p}}.\end{aligned}$$

Hence,

$$|A_{j+1}|^{\frac{p}{p-1}} \le \left(\frac{\gamma}{\alpha}\right)^{\frac{p}{p-1}} |Q_1|^{\frac{1}{p-1}} |A_j \setminus A_{j+1}|.$$

Summing the last inequality over $j = 0, 1, \dots, j_* - 1$, we obtain

$$j_* |A_{j_*}|^{\frac{p}{p-1}} \le \left(\frac{\gamma}{\alpha}\right)^{\frac{p}{p-1}} |Q_1|^{\frac{p}{p-1}},$$

whence

$$|A_{j_*}| \le \frac{\gamma}{\alpha}\left(\frac{1}{j_*}\right)^{\frac{p-1}{p}} |Q_1|.$$

The lemma is proved with

$$j_* = \left(\frac{\gamma}{\varepsilon\alpha}\right)^{\frac{p}{p-1}}, \quad \theta(\varepsilon,\alpha) = 2^{(p-2)j_*}.$$

□

Now we prove the following

Proposition 4.1.13. *Let v be as in Lemma 4.1.12. Denote*

$$Q(\theta) = K_1^0 \times [(2^{p+1} - 1)\theta, 2^{p+1}\theta].$$

There exists a constant $\theta_o = \theta_o(data, \alpha)$ such that

$$\operatorname*{ess\,inf}_{Q(\theta)} v(x,t) \geq \frac{1}{2}\theta^{\frac{1}{2-p}}$$

for all $\theta \geq \theta_o$. Given α, we denote the corresponding value of θ_o by $\theta_o(\alpha)$.

Proof. With reference to Lemma 4.1.9, let $\varepsilon_o = s(\frac{1}{2}, 1)$. Correspondingly let $\theta_o = \theta(\alpha, \varepsilon_o)$ as given by Lemma 4.1.12. Now choose $\varepsilon < \varepsilon_o$ and let $\theta = \theta(\varepsilon, \alpha)$. Then in the cylinder $Q_1(\theta) = K_2^0 \times [2^p\theta, 2^{p+1}\theta]$ we have

$$|\{(x,t) \in Q_1(\theta) : v(x,t) < \theta^{\frac{1}{2-p}}\}| < \varepsilon |Q_1(\theta)|.$$

In the cylinder $Q_1(\theta)$ apply Lemma 4.1.9 with $\mu_- = 0$ and $a = \frac{1}{2}$ to conclude the proof. $\square$

Proposition 4.1.14. *Let u be a non-negative supersolution to (4.1.1) in the cylinder $Q = K_1^0 \times [-1, 0]$. Let*

$$\left|\left\{u \geq \frac{1}{2}\right\} \cap Q\right| \geq \frac{1}{2}$$

and

$$\iint_Q |D(u - \frac{1}{2})_-| dxdt \leq \tilde{\gamma}.$$

Then for any $\sigma \in (0,1)$ there exist $\eta_o = \eta_o(data, \sigma, \gamma) \in (0,1)$ and $(y,s) \in Q$ such that

$$Q_\sigma \equiv K_{\eta_o}^y \times \left[s - \eta_o^p \left(\frac{1}{4}\right)^{2-p}, s\right] \subset Q$$

and

$$\left|\left\{u \geq \frac{1}{4}\right\} \cap Q_\sigma\right| \geq \sigma |Q_\sigma|.$$

Proof. First, we show that there exists $\tau_* \in [-1, -\frac{1}{16}]$ such that

$$\int_{K_1^0} |D(u - \frac{1}{2})_-|(y, \tau_*)\, dy \leq 16\tilde{\gamma} \tag{4.1.13}$$

and

$$\left|\left\{u(y,\tau_*) \geq \frac{1}{2}\right\} \cap K_1^0\right| \geq \frac{3}{8}. \tag{4.1.14}$$

It is obvious that the measure of the subset of $[-1, 0]$ where (4.1.13) does not hold does not exceed $\frac{1}{16}$. Consequently, (4.1.13) holds on a set of measure at least $\frac{15}{16}$. Next, it is easy to see that the set of $\tau \in (-1,0]$ where (4.1.14) does not hold has measure less than $\frac{14}{16}$. Hence, both (4.1.13) and (4.1.14) hold on a set of measure strictly larger than $\frac{1}{16}$.

We apply the result of [27] to $u(\cdot,\tau_*)$ in K_1^0, and deduce that, for any $\bar\sigma \in (0,1)$ there exist $y_1 \in K_1^0$ and $\bar\eta \in (0,1)$ such that

$$\left|\left\{x \in K_{\bar\eta}^{y_1} : u(x,\tau_*) > \frac{3}{8}\right\}\right| > \bar\sigma |K_{\bar\eta}^{y_1}|.$$

Let $\theta = 4^{2-p}$, and set

$$\tau_*' = \tau_* + \left(\frac{1}{4}\right)^{2-p}\left(\frac{\bar\eta}{2}\right)^p (1-\bar\sigma).$$

Consider the cylinder

$$Q_1 = K_{\bar\eta}^{y_1} \times [\tau_*, \tau_*'].$$

Writing the energy inequality (4.1.9) over the cylinder Q_1 with $k = \frac{3}{8}$ and a proper cut–off function, we obtain

$$\max_{\tau_* \le t \le \tau_*'} \int_{K_{\bar\eta/2}^{y_1}} \left(u - \frac{3}{8}\right)_-^2 dx \le \gamma \left((1-\bar\sigma)\left(\frac{3}{8}\right)^2 |K_{\bar\eta}^{y_1}| \right.$$
$$\left. + \left(\frac{4}{\bar\eta}\right)^p \left(\frac{3}{8}\right)^p |K_{\bar\eta}^{y_1}| \left(\frac{1}{4}\right)^{2-p}\left(\frac{\bar\eta}{2}\right)^p (1-\bar\sigma)\right).$$

Therefore, $\forall\, t \in [\tau_*, \tau_*']$

$$|\{x \in K_{\bar\eta/2}^{y_1} : u(x,t) \le \frac{1}{4}\}| \le 64\gamma_1(1-\bar\sigma)\,|K_{\bar\eta/2}^{y_1}|.$$

If we take $\bar\sigma = 1 - \dfrac{\sigma 2^{-p}}{64\gamma_1}$, then in the cylinder $Q_2 = K_{\bar\eta/2}^{y_1} \times [\tau_*, \tau_*']$ we obtain

$$|\{u \le \frac{1}{4}\} \cap Q_2| < \sigma 2^{-p} |Q_2|.$$

Up to a zero measure set, decompose the base of Q_2 into the 2^{ln} congruent cubes $K_{2^{-l-1}\bar\eta}^{z_j}$, $j = 1, \ldots, 2^{ln}$. Choose the smallest natural number l such that

$$(2^{-l-1}\bar\eta)^p \le \left(\frac{\bar\eta}{2}\right)^p (1-\bar\sigma).$$

There exists at least j such that in the cylinder $\tilde Q_j = K_{2^{-l-1}\bar\eta}^{z_j} \times (\tau_*, \tau_*']$ we have

$$|\{u \le \frac{1}{4}\} \cap \tilde Q_j| < \sigma 2^{-p} |\tilde Q_j|.$$

In the cylinder

$$Q_\sigma = K^{z_j}_{2^{-l-1}\bar\eta} \times \left[\tau_*, \tau_* + (2^{-l-1}\bar\eta)^p \left(\frac{1}{4}\right)^{2-p}\right]$$

we have

$$|\{u \le \frac{1}{4}\} \cap Q_\sigma| < \sigma |Q_\sigma|.$$

□

We will use the following corollary

Corollary 4.1.15. *Let u be as in Proposition 4.1.14. Then there exist a number $\eta_o > 0$ and a cylinder*

$$Q_3 = K^{y_o}_{\eta_o} \times [s_o - \left(\frac{\eta_o}{2}\right)^p \left(\frac{1}{4}\right)^{2-p}, s_o] \subset K^0_1 \times [-1, 0]$$

such that

$$\operatorname*{ess\,inf}_{Q_3} u \ge \frac{1}{8}.$$

Proof. Take $\sigma = s(\frac{1}{2}, 1)$ and apply Lemma 4.1.9 in the cylinder Q_σ constructed in Proposition 4.1.14 with $\mu_- = 0$, $\omega = \frac{1}{4}$, $a = \frac{1}{2}$. □

4. Proof of Lemma 4.1.2

First, from the energy estimate (4.1.9) it follows that

$$\iint_{K^0_1 \times [-1,0]} |D(u - \frac{1}{2})_-| \, dxdt \le \gamma(data).$$

Next, use Corollaries 4.1.15 and 4.1.11 to obtain $y_o \in K^0_1$, $\eta_o \in (0,1)$ and $t_o \in [-\frac{1}{16}, 0]$ such that

$$\operatorname*{ess\,inf}_{K^{y_o}_{\eta_o/2}} u(x,t) \ge \frac{1}{16}\left(1 + \frac{t - t_o}{\nu 8^{p-2}\eta_o^p}\right)^{\frac{1}{2-p}}, \quad t \ge t_o,$$

where $\eta_o = \eta_o(data) > 0$. Hence,

$$\operatorname*{ess\,inf}_{K^{y_o}_{\eta_o/2}} u(x,0) \ge \mu_o := \frac{1}{16}\left(1 + \frac{1}{\nu 8^{p-2}\eta_o^p}\right)^{\frac{1}{2-p}}.$$

Apply Corollary 4.1.11 again to obtain that

$$\operatorname*{ess\,inf}_{K^{y_o}_{\eta_o/4}} u(x,t) \ge \psi(t) := \frac{\mu_o}{2}\left(1 + \frac{t}{\nu \mu_o^{2-p}(\eta_o/2)^p}\right)^{\frac{1}{2-p}}, \quad t \ge 0.$$

Change the variables in equation (4.1.1) as

$$u(x,t) = v(x,t)\psi(t), \quad t = t(\tau),$$

where τ is a solution to the problem

$$\frac{d\tau}{dt} = \psi^{p-2}(t), \quad \tau(0) = 0.$$

One can see that

$$\tau(t) = \frac{\nu}{2^{p-2}}\left(\frac{\eta_o}{2}\right)^p \ln\left(1 + \frac{t}{\nu\mu_o^{2-p}(\eta_o/2)^p}\right),$$

$$\psi(t(\tau)) = \frac{\mu_o}{2}\exp\left[\frac{2^{p-2}\tau}{(2-p)\nu(\eta_o/2)^p}\right],$$

and for all $\tau \geq 0$ we have

$$\operatorname*{ess\,inf}_{K^{y_o}_{\eta_o/4}} v(x,\tau) \geq 1.$$

It can be verified that v is a supersolution to the equation

$$v_\tau = \operatorname{div}\mathbf{A}_1(x,\tau,v,Dv),$$

where

$$\mathbf{A}_1(x,\tau,v,Dv) = \psi^{1-p}(t)\mathbf{A}(x,t,\psi v,\psi Dv).$$

Moreover, $\mathbf{A}_1$ satisfies the same structural conditions as $\mathbf{A}$. Applying Proposition 4.1.13 we obtain that

$$\operatorname*{ess\,inf}_{K^0_1} v(x,\tau) \geq \frac{1}{2}\theta^{\frac{1}{2-p}}$$

with $\theta = \theta_o((\frac{\eta_o}{4})^n)$ for all $\tau \in [(2^{p+1}-1)\theta, 2^{p+1}\theta]$.

Returning to the original variables we see that

$$\operatorname*{ess\,inf}_{K^0_1} u(x,t) \geq \mu := \frac{1}{2}\theta^{\frac{1}{2-p}}\frac{\mu_o}{2}\exp\left[\frac{2^{2p-1}\theta}{(2-p)\nu(\eta_o/2)^p}\right]$$

for all $t \in [\gamma_1, \gamma_2]$ where we have set

$$\gamma_1 = \nu\mu_o^{2-p}\left(\frac{\eta_o}{2}\right)^p\left[\exp\left(\frac{2^{p-2}(2^{p+1}-1)\theta}{\nu(\eta_o/2)^p}\right) - 1\right],$$

$$\gamma_2 = \nu\mu_o^{2-p}\left(\frac{\eta_o}{2}\right)^p\left[\exp\left(\frac{2^{2p-1}\theta}{\nu(\eta_o/2)^p}\right) - 1\right].$$

$\square$

Remark. One can see that the constant μ, and consequently, the Hölder constants α_1 and α_2 deteriorate as $p \to 2$. Indeed it can be shown that these constants can be stabilized. One only needs to repeat the argument of Lemma 7.1 of [28] with obvious modifications.

4.2 A Harnack inequality for weighted degenerate parabolic equations.

This section follows line by line my article [106].

Let Ω be a bounded domain in $\mathbb{R}^n$. Denote $\Omega_T = \Omega \times [T_1, T_2] \subset \mathbb{R}^{n+1}$, $n \geq 2$. In the cylinder Ω_T we consider the equation

$$u_t - \operatorname{div} \mathbf{A}(x,t,u,Du) = B(x,t,u,Du). \tag{4.2.1}$$

We assume that $\mathbf{A}(x,t,u,\mathbf{p}) : \Omega_T \times \mathbb{R}^{n+1} \to \mathbb{R}^n$, $B(x,t,u,\mathbf{p}) : \Omega_T \times \mathbb{R}^{n+1} \to \mathbb{R}$ are Carathéodory functions satisfying the following structure conditions:

$$\mathbf{A}(x,t,u,Du) \cdot Du \geq C_0 \nu(x) |Du|^p - C^p \nu(x), \tag{4.2.2}$$
$$|\mathbf{A}(x,t,u,Du)| \leq C_1 \nu(x) |Du|^{p-1} + C^{p-1} \nu(x), \tag{4.2.3}$$
$$|B(x,t,u,Du)| \leq C_2 \nu(x) |Du|^{p-1} + C^{p-1} \nu(x) \tag{4.2.4}$$

almost everywhere (a.e.) in Ω_T. Here $p = const > 2$, the constants C_0, C_1 are positive and the constants C_2, C are nonnegative.

We assume that the function $\nu \in A_{1+\frac{p}{n}}$, where $A_{\dots}$ denotes the Muckenhoupt class. This means that

$$C_\nu := \sup \left(\frac{1}{|K|} \int_K \nu(x) dx \right) \left(\frac{1}{|K|} \int_K \left(\frac{1}{\nu(x)} \right)^{n/p} dx \right)^{p/n} < +\infty, \tag{4.2.5}$$

where the supremum is taken over all cubes $K \subset \mathbb{R}^n$ with faces parallel to the coordinate planes. It is not hard to see that $|x|^\alpha \in A_{1+\frac{p}{n}}$ if $-n < \alpha < p$.

For a bounded open set E by $W^{1,p}(E,\nu)$ we denote the closure of $C^\infty(E)$ with respect to the norm

$$\|\phi\|_{W^{1,p}(E,\nu)} = \left(\int_E (|\phi|^p + |D\phi|^p) \, \nu dx \right)^{1/p}.$$

For a bounded set K by $W_0^{1,p}(K,\nu)$ we denote the closure of $C_0^\infty(K)$ with respect to the norm

$$\|\phi\|_{W_0^{1,p}(K,\nu)} = \left(\int_K |D\phi|^p \, \nu dx \right)^{\frac{1}{p}}.$$

We say that u is a super(sub)-solution to equation (4.2.1) in Ω_T if

$$u \in C([T_1,T_2]; L^2(\Omega)) \cap L^p([T_1,T_2]; W^{1,p}(\Omega,\nu))$$

and for any $[t_1,t_2] \subset [T_1,T_2]$ and any nonnegative

$$\xi \in W^{1,2}([t_1,t_2]; L^2(\Omega)) \cap L^p([t_1,t_2]; W_0^{1,p}(\Omega,\nu))$$

we have

$$\int_\Omega u\xi dx\Big|_{t_1}^{t_2} - \iint_{\Omega\times[t_1,t_2]} u\xi_t dxdt + \iint_{\Omega\times[t_1,t_2]} \mathbf{A}(x,t,u,Du)\cdot D\xi dxdt$$
$$\geq (\leq) \iint_{\Omega\times[t_1,t_2]} B(x,t,u,Du)\xi dxdt$$

We say that u is a solution if it is both supersolution and subsolution.

We say that a constant γ depends on the data ($\gamma = \gamma(data)$) if it can be quantitatively expressed via $C_0, C_1, C_2, n, p, C_\nu, diam\Omega$. We say that a constant c depends on the weight ν ($c = c(\nu)$) if it can be quantitatively expressed via n, p, C_ν.

For $y \in \mathbb{R}^n$ and $\rho > 0$ we denote

$$K_\rho^y = \{x \in \mathbb{R}^n : |x_i - y_i| < \rho/2,\ i = 1, \dots n\}.$$

For a measurable set $E \subset \mathbb{R}^n$ and a nonnegative $\omega \in L^1_{loc}(\mathbb{R}^n)$ we denote $\omega(E) = \int_E \omega(x)dx$. For a set $E \subset \mathbb{R}^{n+1}$ we denote $\omega(E) = \int_E \omega(x)dxdt$. For a cylinder $Q = K_\rho^y \times [t_0 - \theta h(y,\rho), t_0]$ and a positive number σ by σQ we denote the cylinder $\sigma Q = K_{\sigma\rho}^y \times [t_0 - \theta h(y,\sigma\rho), t_0]$. When we speak about nonnegative (sub, super-) solutions we understand it in the sense of a.e.

Following [18], where the case $p = 2$ was studied, we introduce the function

$$h(y,\rho) = \left(\int_{K_\rho^y} \nu^{-n/p} dx\right)^{p/n}.$$

It is easy to see that in case $\nu \equiv 1$ we have $h(x,\rho) = \rho^p$. Moreover, the definition of the Muckenhoupt class $A_{1+\frac{p}{n}}$ immediately yields the following useful relation:

$$\rho^{n+p} \leq \nu(K_\rho^x)h(x,\rho) \leq C_\nu \rho^{n+p}, \tag{4.2.6}$$

where the first inequality follows immediately from the Hölder inequality. We often use the obvious consequence of this relation: let $Q = K_\rho^x \times [t_1, t_2]$. Then

$$\frac{\rho^p|Q|}{h(x,\rho)} \leq \nu(Q) \leq C_\nu \frac{\rho^p|Q|}{h(x,\rho)}. \tag{4.2.7}$$

Observe that for the cylinder $Q = K_\rho^x \times [t_0, t_0 + \theta h(x,\rho)]$ we have $\theta\rho^{n+p} \leq \nu(Q) \leq C_\nu \theta \rho^{n+p}$, i.e. the ν–measure of Q is comparable with the euclidian measure of the standard parabolic cylinder.

The letter Q (with various sub- and superindices) will be used to refer to a cylinder and the letter K to refer to a cube. We will use γ, γ_1, γ_2, ... for the constants which depend on the data and c, c_1, c_2, ... for the constants which depend on the weight ν. The exact value of the constants γ and c varies from line to line but in each case it is clear from the context.

Now we are ready to formulate the main result of the paper.

Theorem 4.2.1. *Let u be a nonnegative solution to equation (4.2.1) in Ω_T. Let $(x_0, t_0) \in \Omega_T$. Denote $Q_\tau = K^{x_0}_{\tau\rho} \times [t_0 - \tau h(x_0, \rho), t_0]$ and let $k > 0$ be such that*

$$k \leq \lim_{\tau \to 0+} \operatorname*{ess\,sup}_{Q_\tau} u(x,t).$$

Let θ be a positive constant. There exist positive constants $\Lambda_1, \Lambda_2, \Lambda_3$ such that if the cylinder $K^{x_0}_{25\rho} \times [t_0 - \theta h(x_0, \rho) k^{2-p}, t_0 + \Lambda_1 h(x_0, \rho) k^{2-p}] \subset \Omega_T$ and $k \geq \Lambda_3 C\rho$ then

$$\operatorname*{ess\,inf}_{x \in K^{x_0}_\rho} u(x, t_0 + \Lambda_1 h(x_0, \rho) k^{2-p}) \geq \Lambda_2 k.$$

The constants Λ_1, Λ_2 and Λ_3 depend on the data and θ only.

The proof closely follows the scheme of [28]. The proof of the next theorem is a direct consequence of the main result.

Theorem 4.2.2. *Let u be a solution to equation (4.2.1) in Ω_T. Then there exists $\hat{u} \in C(\Omega \times (T_1, T_2])$ which coincides with u almost everywhere in Ω_T.*

We also need the following result.

Theorem 4.2.3. *Let u be a solution of (4.2.1) in the cylinder $Q = K^{x_0}_\rho \times [t_0 - \theta h(y, \rho), t_0]$, where θ is a positive constant. Then for any $\sigma \in (0,1)$*

$$\operatorname*{ess\,sup}_{Q_\sigma} |u(x,t)| < \infty,$$

where $Q_\sigma = K^{x_0}_{\sigma\rho} \times [t_0 - \theta\sigma h(y, \rho), t_0]$.

The quantitative bound is contained in the proof.

Some remarks on the history of the question. The Harnack inequality for linear parabolic equations in divergent form is known since the seminal works of J. Moser ([89], [90], [91]). Moser's results were almost immediately generalised to the quasilinear case by D.G. Aronson and J. Serrin in [4] and N.S. Trudinger in [108].

Analogous results for the parabolic p-Laplace type equations appeared much later. For the equation $u_t = \Delta_p u$ the Harnack inequality was proved in [24] (also [26]). Despite the fact that (technically) relatively close result on the Hölder continuity was proved by E. DiBenedetto in 1980's , the proof of the Harnack inequality for the parabolic p-Laplace type equations with general structure conditions lacked until recently. The well-known book [25] contains a relatively complete account of the state of the art in the field by the beginning of 1990's. The survey article [31] contains a very clear exposition of the ideas and techniques used in nonlinear parabolic regularity theory and an updated bibliography.

In the breakthrough paper [28] the Harnack inequality was finally proved for the case $p > 2$. The same authors also presented the proof for the case $p < 2$ ([29]). The main part of the proof was the 'expansion of positivity' argument, which is known in a different form for linear equations since the works of E.M. Landis ([82]). It is also worth mentioning the earlier works [45], [44] and [27], which contain some important ideas that constituted the proof in [28]. Slightly later, in [78](see also [79]) Tuomo Kuusi presented a proof based on the different approach, somewhat similar to the one used by Krylov and Safonov in their celebrated paper [77]. In [40] S. Fornaro and M.Sosio considered the class of doubly nonlinear degenerate parabolic equations using the techniques developed in [28].

In all these cases the critical feature, distinguishing between the nonlinear case and the linear one, is the presence of the 'intrinsic scaling' effect, i.e. the dependence of the size of the 'natural' parabolic cylinder on the value of the solution. This effect and overcoming its consequences usually presents the most delicate part of the proof. Moreover, the standard Moser-type arguments (Parabolic BMO or the Bombieri lemma) cease to be applicable.

The nonuniformly elliptic and nonuniformly parabolic equations have been studied for a long time. The first results for elliptic equations which allowed for a sufficiently wide class of weights were obtained in [37] for linear elliptic equations. This paper attracted a great deal of attention to the subject and induced many follow ups both for the elliptic and the parabolic cases. The analogue of the result of [37] (a priori bounds, existence, Harnack inequality, continuity) for the linear parabolic equations was obtained in [15], [18]. In these papers the weight was assumed to belong to the Muckenhoupt class $A_{1+\frac{2}{n}}$. In [19] and [17] this result was generalised to the case of a time-dependent weights $\nu(x,t)$ satisfying certain Muckenhoupt-type conditions. In the paper [20] the same authors proved an interesting parallel result for a class of equations of the type $\nu(x)u_t = \nabla\nu(x)\nabla u$, $\nu(x) \in A_2$. The results of Chiarenza and Serapioni were generalised for a very general framework in [50], [51], [38]. All the papers cited above employ the Moser's method.

In [2] the Harnack inequality was obtained for the equation

$$u_t = \nabla(|x|^{-p\gamma}|\nabla|^{p-2}\nabla u)$$

with $p > 2$ for certain values of γ. The proof in this paper utilizes the old scheme of DiBenedetto [24] which in turn relies on the existence of explicit sub- and supersolutions. Thus, it is not applicable for a weight of the general form.

In this paper we use the new approach developed in [28]. Only the case of the time-independent weight is considered. The condition on the weight repeats the condition of [18] for $p \neq 2$. The parabolic cylinders, natural for the equation, depend both on the solution and on the position of the cylinder. The next interesting step in the direction would be prove the corresponding results for a time-dependent weight $\nu(x,t)$.

We also note that the result of our paper covers the result of [18]. The proof in this case is simplified since the effect of the intrinsic scaling disappears. One simply repeats all the arguments with $p = 2$ except the change of variables used to prove the 'expansion of positivity' result, which is no more needed.

Muckenhoupt classes and their properties. In this section for convenience of the reader we collected the properties of the Muckenhoupt classes we use in this paper. By definition, $\omega \in A_q$ if ω is a nonnegative locally integrable function such that

$$C_{q,\omega} := \sup \left(\frac{1}{|K|}\int_K \omega(x)dx\right)\left(\frac{1}{|K|}\int_K (\omega(x))^{\frac{1}{1-q}}\,dx\right)^{q-1} < +\infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ with faces parallel to the coordinate planes. In the following, only such cubes will be considered. We make here the following obvious observation. If $\omega \in A_q(\mathbb{R}^n)$ then for $\omega_1(x,t) = \omega(x)$ we have $\omega_1 \in A_q(\mathbb{R}^{n+1})$.

1. (*Fairness*) Let K be a cube and $E \subset K$. Let $\omega \in A_q$. As an almost immediate consequence of the definition of A_q one has

$$\left(\frac{|E|}{|K|}\right)^q \leq C_{q,\omega}\frac{\omega(E)}{\omega(K)}. \tag{4.2.8}$$

Moreover, there exist constants $\kappa \in (0,1]$ and $c_1 > 0$ such that

$$\frac{\omega(E)}{\omega(K)} \leq c_1\left(\frac{|E|}{|K|}\right)^\kappa. \tag{4.2.9}$$

The constants κ and c_1 depend only on $n, q, C_{q,\omega}$. Note that (4.2.8) immediately implies the doubling property: for any cube $K \in \mathbb{R}^n$ we have $\omega(2K) \leq c_3\omega(K)$, where $2K$ denotes the cube with the same center as K and twice the length of the edge.

2. If $\omega \in A_q$ then $\omega^{\frac{1}{1-q}} \in A_{q'}$, where $\frac{1}{q} + \frac{1}{q'} = 1$. It follows that for $\omega^{1/(1-q)}$ we also have the relations like (4.2.8), (4.2.9). In the context of this work, we note that $\nu^{-n/p} \in A_{1+\frac{n}{p}}$. We derive from here some useful consequences.

First, $\nu^{-n/p}$ satisfies the doubling condition: there exists a constant $c = c(\nu)$ such that $h(y, 2\rho) \leq ch(y,\rho)$ for all $y \in \mathbb{R}^n$.

Second, the property (4.2.9) applied to $\nu^{-n/p}$ gives a constant $\kappa = \kappa(\nu) \in (0, p]$ such that for any $x \in \mathbb{R}^n$, $r > 0$ and $\varepsilon \in (0, 1)$ we have

$$h(x, \varepsilon r) \leq c\varepsilon^{\kappa} h(x, r), \tag{4.2.10}$$

where $c = c(\nu)$.

Third, the property (4.2.8) applied to $\nu^{-n/p}$ implies that for any $x \in \mathbb{R}^n$, $r > 0$ and $\varepsilon \in (0, 1)$

$$h(x, \varepsilon r) \geq c\varepsilon^{p+n} h(x, r), \tag{4.2.11}$$

where $c = c(\nu)$.

3. (*Reverse Hölder inequality*). Let $\omega \in A_q$. Then there exist constants $\delta > 0$ and $c > 0$, which depend only on n, q and $C_{q,\omega}$, such that for any cube $K \in \mathbb{R}^n$ we have

$$\frac{1}{|K|}\int_K \nu^{1+\delta} dx \leq \left(\frac{1}{|K|}\int_K \nu dx\right)^{1+\delta}. \tag{4.2.12}$$

4. (*Open-End property*). Let $\omega \in A_q$, $q > 1$. Then $\omega \in A_{q'}$ with $q' < q$. The values of q' and $C_{q',w}$ depend on $q, C_{q,\omega}, n$. This property is a consequence of the reverse Hölder inequality.

Sobolev-Gagliardo-Nirenberg type inequalities. Let us recall first the well-known result which was first proved in [37]. Its simple proof can be found in [16].

Lemma 4.2.4. *Let $q \in (p/n, p]$ be such that $\omega \in A_q$. Then for any cube $K = K_\rho^y \subset \mathbb{R}^n$ and any function $v \in C_0^\infty(K)$ we have*

$$\frac{1}{\omega(K)}\int_K |v|^{pk}\omega dx \leq \gamma\rho^{pk}\left(\frac{1}{\omega(K)}\int_K |Dv|^p\omega dx\right)^k,$$

where $k = \frac{nq}{nq-p}$ and the constant γ depends only on n, p, $C_{q,\omega}$.

Our proof of the inequalities required in the parabolic case follows the line of [18], [17].

Lemma 4.2.5. *There exists a constant $h_0 = h_0(\nu)$ such that for any $h \in (1, h_0)$, any cube $K = K_\rho^y \subset \mathbb{R}^n$ and any function $v \in C_0^\infty(K)$ we have*

$$\frac{1}{\nu(K)}\int_K |v|^{ph}\nu dx \leq c\left(\frac{1}{|K|}\int_K |v|^p dx\right)^{h-1}\frac{\rho^p}{\nu(K)}\int_K |Dv|^p\nu dx,$$

where $c = c(\nu, h)$.

Proof. Apply the Hölder inequality, the inequality of Lemma 4.2.4, and the reverse Hölder inequality (4.2.12) to estimate

$$\int_K |v|^{ph}\nu dx \le \left(\int_K |v|^{pk}\nu dx\right)^{\frac{1}{k}} \left(\int_K \nu^{1+\delta}dx\right)^{\gamma} \left(\int_K |v|^p dx\right)^{h-1}$$
$$\le \nu(K)^{1/k}|K|^{h-1+\gamma}\left(\frac{\nu(K)}{|K|}\right)^{(1+\delta)\gamma}\left(\rho^{pk}\left(\frac{1}{\nu(K)}\int_K |Dv|^p\nu dx\right)^k\right)^{1/k}$$
$$\times\left(\frac{1}{|K|}\int_K |v|^p dx\right)^{h-1}.$$

For the Hölder inequality to hold, the coefficients must be related as

$$\frac{1}{k}+\gamma+h-1=1.$$

To eliminate $|K|$ on the right-hand side we need

$$h-1+\gamma-(1+\delta)\gamma = h-1-\delta\gamma = 0.$$

Hence,

$$\gamma = \frac{h-1}{\delta}.$$

Further, we need $\nu(K)$ on the right-hand side to get the desired inequality. Calculating the power we have

$$\frac{1}{k}+(h-1)\frac{1+\delta}{\delta} = 1$$

From the last equation we find

$$h = 1+\frac{\delta}{1+\delta}\cdot\frac{p}{nq}.$$

The number δ here is chosen such that ν satisfies the reverse Hölder inequality (4.2.12). The number q here can be any number from the interval $(\frac{p}{n}, p)$ such that $\nu \in A_q$. □

Lemma 4.2.6. *There exists a constant $h_0 = h_0(\nu)$ such that for any $h \in (1, h_0)$, any cube $K = K^y_\rho \subset \mathbb{R}^n$ and any function $v \in C_0^\infty(K)$ we have*

$$\frac{1}{|K|}\int_K |v|^{ph}dx \le c\left(\frac{1}{|K|}\int_K |v|^p dx\right)^{h-1}\frac{\rho^p}{\nu(K)}\int_K |Dv|^p\nu dx,$$

where $c = c(\nu,\ h)$.

Proof. By the open-end property of Muckenhoupt weights, there exists such $q > \frac{n}{p}$ that $\nu \in A_{1+\frac{1}{q}}$. Apply successively Hölder inequality, the inequality of Lemma 4.2.4 and the definition of $A_{1+\frac{1}{q}}$ to obtain

$$\int_K |v|^{ph} dx \leq \left(\int_K |v|^{pk} \nu dx\right)^{\frac{1}{k}} \left(\int_K |v|^p dx\right)^{h-1} \left(\int_K \left(\frac{1}{\nu}\right)^q dx\right)^{\frac{1}{kq}}$$

$$\leq \nu(K)^{\frac{1}{k}} |K|^{h-1+\frac{1}{kq}} \left(\frac{1}{|K|}\int_K \nu dx\right)^{-\frac{1}{k}} \frac{\rho^p}{\nu(K)} \int_K |Dv|^p \nu dx \left(\frac{1}{|K|}\int_K |v|^p dx\right)^{h-1}.$$

Here $k = \frac{nq_1}{nq_1 - p}$, where $q_1 \in (\frac{p}{n}, p)$ is such that $\nu \in A_{q_1}$. For the Hölder inequality to hold, the coefficients k, h, and q must satisfy

$$\frac{1}{k} + h - 1 + \frac{1}{kq} = 1.$$

Hence, we find

$$h = 1 + \left(1 - \frac{1}{k}\left(1 + \frac{1}{q}\right)\right).$$

To get $k > 1$ we must take

$$k > 1 + \frac{1}{q}. \tag{4.2.13}$$

Take $q_1 = 1 + \frac{1}{q}$. Then the inequality (4.2.13) is satisfied. Note also that for $n \geq 2$ and $p \geq 2$ always $1 + \frac{p}{n} \leq p$. Thus, we obtained the desired inequality with

$$h = 1 + \frac{p}{n} - \frac{1}{q} > 1. \quad \square$$

To prove the local boundedness of solutions we need the variant of two preceding lemmas.

Lemma 4.2.7. *There exists a constant $h_0 = h_0(\nu)$ such that for any $h \in (1, h_0)$, any cube $K = K_\rho^y \subset \mathbb{R}^n$ and any function $v \in C_0^\infty(K)$ we have*

$$\frac{1}{\nu(K)} \int_K |v|^{ph} \nu dx \leq c \left(\frac{1}{|K|}\int_K |v|^2 dx\right)^{h-1} \frac{\rho^p}{\nu(K)} \int_K |Dv|^p \nu dx,$$

where $c = c(\nu, h)$.

Proof. Let $q \in (\frac{p}{n}, p]$ be such that $\nu \in A_q$. Denote $k = \frac{nq}{nq-p}$. Using successively the Hölder inequality, the inequality of Lemma 4.2.4 and the reverse Hölder inequality (4.2.12), estimate

$$\int_K |v|^{ph} \nu dx \leq \left(\int_K |v|^{pk} \nu dx\right)^{\frac{1}{k}} \left(\int_K \nu^{1+\delta} dx\right)^{\gamma} \left(\int_K |v|^2 dx\right)^{\frac{p(h-1)}{2}}$$

$$\leq \nu(K)^{\frac{1}{k}} |B|^{\frac{p(h-1)}{2}+\gamma} \left(\frac{\nu(K)}{|K|}\right)^{(1+\delta)\gamma} \frac{\rho^p}{\nu(K)} \int_K |Dv|^p \nu dx \left(\frac{1}{|K|}\int_K |v|^2 dx\right)^{\frac{p(h-1)}{2}}.$$

To get the desired result, we need that the coefficients satisfy the equations

$$\frac{1}{k} + \gamma + \frac{p(h-1)}{2} = 1 \quad \textit{(Hölder)},$$

$$\frac{p(h-1)}{2} + \gamma - (1+\delta)\gamma = 0 \quad \textit{(balance w.r.t. } |K|\textit{)},$$

$$\frac{1}{k} + (1+\delta)\gamma = 1 \quad \textit{(balance w.r.t. } \nu(K)\textit{)}.$$

From the second relation we find $\gamma = \frac{p(h-1)}{2\delta}$. Substituting it into the third relation we find

$$h = 1 + \frac{k-1}{k} \cdot \frac{2}{p} \cdot \frac{\delta}{1+\delta} = 1 + \frac{2\delta}{nq(1+\delta)}. \quad \square$$

Lemma 4.2.8. *There exists a constant $h_0 = h_0(\nu)$ such that for any $h \in (1, h_0)$, any cube $K = K^y_\rho \subset \mathbb{R}^n$ and any function $v \in C_0^\infty(K)$*

$$\frac{1}{\nu(K)} \int_K |v|^{ph} dx \leq c \left(\frac{1}{|K|} \int_K |v|^2 dx \right)^{\frac{p(h-1)}{2}} \frac{\rho^p}{\nu(K)} \int_K |Dv|^p \nu dx,$$

where $c = c(\nu, h)$.

Proof. Choose $q > \frac{n}{p}$ such that $\nu \in A_{1+\frac{1}{q}}$. Let $q_1 \in (\frac{p}{n}, p]$ be such that $\nu \in A_{q_1}$. Denote $k = \frac{nq_1}{nq_1 - p}$. Estimate

$$\begin{aligned}
\int_K |v|^{ph} dx &\leq \left(\int_K |v|^{pk} \nu dx \right)^{\frac{1}{k}} \left(\int_K |v|^2 dx \right)^{\frac{p(h-1)}{2}} \left(\int_K \left(\frac{1}{\nu}\right)^q dx \right)^{\frac{1}{kq}} \\
&\leq c\nu(K)^{\frac{1}{k}} |K|^{\frac{p(h-1)}{2}} |K|^{\frac{1}{kq}} \left(\frac{1}{|K|} \int_K \nu dx \right)^{-\frac{1}{k}} \\
&\quad \times \frac{\rho^p}{\nu(K)} \int_K |Dv|^p \nu dx \left(\frac{1}{|K|} \int_K |v|^2 dx \right)^{\frac{p(h-1)}{2}},
\end{aligned}$$

where

$$\frac{1}{k} + \frac{p(h-1)}{2} + \frac{1}{kq} = 1.$$

Hence,

$$h = 1 + \frac{2}{p} \left(1 - \frac{1}{k} \left(1 + \frac{1}{q} \right) \right).$$

To get $h > 1$ we must take $k > 1 + \frac{1}{q}$. Choose $q_1 = 1 + \frac{1}{q}$. Then

$$\frac{k}{q_1} = \frac{n}{nq_1 - p} = \left(1 + \frac{1}{q} - \frac{p}{n} \right)^{-1} > 1.$$

It is easy to see that with this choice of parameters we obtain the desired inequality with

$$h = 1 + \frac{2}{p}\left(\frac{p}{n} - \frac{1}{q}\right) = 1 + \frac{2}{n} - \frac{2}{pq}.$$

□

Remark. In fact, we use the result of the last lemma in the following form:

$$\frac{1}{|K|}\int_K |v|^{2\frac{ph}{2}} dx \le c\left(\frac{1}{|K|}\int_K |v|^2 dx\right)^{\frac{p}{2}(h-1)} \frac{\rho^p}{\nu(K)}\int_K |Dv|^p \nu dx.$$

For the proof of the Harnack inequality we briefly summarize this results in the form of

Corollary 4.2.9. *There exist constants c and $h_0 > 1$ such that for any $h \in (1, h_0)$, any cube $K = K_\rho^y \Subset \mathbb{R}^n$ and for any $\phi \in W_0^{1,p}(K, \nu)$ we have*

$$\frac{1}{\nu(K)}\int_K |\phi|^{ph} \nu dx \le c\left(\frac{1}{|K|}\int_K |\phi|^p dx\right)^{h-1} \times \frac{\rho^p}{\nu(K)}\int_K |D\phi|^p \nu dx, \tag{4.2.14}$$

$$\frac{1}{|K|}\int_K |\phi|^{ph} dx \le c\left(\frac{1}{|K|}\int_K |\phi|^p dx\right)^{h-1} \times \frac{\rho^p}{\nu(K)}\int_K |D\phi|^p \nu dx. \tag{4.2.15}$$

The detailed description of the Muckenhoupt classes can be found in [53], [102]. The papers [13] and [14] provide useful information on the properties of weighted Sobolev spaces as well as some interesting (counter)examples.

First, we prove Theorem 4.2.1 assuming that the local boundedness of solutions is already known. We postpone the proof of the latter result until the end of the paper.

Remark. Before commencing the proof we make the following remark. We often prove that $\operatorname{ess\,inf}_Q u \ge k$ where $Q = K \times [t_1, t_2]$ and then immediately pass to the conclusion that for all $t \in [t_1, t_2]$ we have $\operatorname{ess\,inf}_{x\in K} u(x,t) \ge k$. This step is justified due to our definition of (sub, super) solution — we have $u \in C([t_1, t_2], L_2(K))$ which implies $(u-k)_- \in C([t_1, t_2], L_2(K))$. Thus, $\int_Q (u-k)_-^2 dxdt = 0$ implies $\int_K (u-k)_-^2(x,t)dx = 0$ for all $t \in [t_1, t_2]$.

Energy estimates. Arguing as in [81] or [25] one can easily get the following family of inequalities. Let the cylinder $Q = K_\rho^y \times [t_1, t_2] \subset \Omega_T$ and ξ be a nonnegative piecewise-smooth function vanishing on the parabolic boundary of

Q. Let u be a subsolution to the equation (4.2.1) in Ω_T. Then for any $k \in \mathbb{R}$ we have

$$\int_{K_\rho^y} (u-k)_+^2 \xi^p dx \Big|_{t_1}^{t_2} + C_0 \iint_Q |D(u-k)_+\xi|^p \nu(x) dxdt$$
$$\leq \tilde{\gamma}_0 \iint_Q (u-k)_+^2 \xi^{p-1}\xi_t dxdt + \tilde{\gamma}_1 \iint_Q (u-k)_+^p |D\xi|^p \nu(x) dxdt$$
$$+\tilde{\gamma}_2 \iint_Q (u-k)_+^p \xi^p \nu(x) dxdt + \tilde{\gamma}_3 \iint_Q \chi_{\{u>k\}} \xi^p \nu(x) dxdt. \tag{4.2.16}$$

If u is a supersolution to (4.2.1) in Ω_T then we have

$$\int_{K_\rho^y} (u-k)_-^2 \xi^p dx \Big|_{t_1}^{t_2} + C_0 \iint_Q |D(u-k)_-\xi|^p \nu(x) dxdt$$
$$\leq \tilde{\gamma}_0 \iint_Q (u-k)_-^2 \xi^{p-1}\xi_t dxdt + \tilde{\gamma}_1 \iint_Q (u-k)_-^p |D\xi|^p \nu(x) dxdt$$
$$+\tilde{\gamma}_2 \iint_Q (u-k)_-^p \xi^p \nu(x) dxdt + \tilde{\gamma}_3 \iint_Q \chi_{\{u<k\}} \xi^p \nu(x) dxdt. \tag{4.2.17}$$

The constant $\tilde{\gamma}_0 = \tilde{\gamma}_0(n,p)$, $\tilde{\gamma}_3 = \tilde{\gamma}_0 C^p$, and

$$\tilde{\gamma}_1 = \tilde{\gamma}_0 \left(1 + \frac{(p-1)^{p-1} C_1^p}{C_0^{p-1}}\right), \quad \tilde{\gamma}_2 = \tilde{\gamma}_0 \left(1 + \frac{(p-1)^{p-1} C_2^p}{C_0^{p-1} p^p}\right).$$

De Giorgi type lemma. For the sake of convenience we formulate this lemma for subsolutions and supersolutions separately.

Lemma 4.2.10. *Let u be a subsolution to (4.2.1) in the cylinder $Q = K_{2\rho}^y \times [t_1 - \theta h(y,2\rho), t_1]$. Let $\mu_+ \geq \operatorname{ess\,sup}_Q u(x,t)$. Then for any $\omega > 0$ and $a \in (0,1)$ there exist numbers s_1^ν and s_1, which depend only on the data, a, and $\theta\omega^{p-2}$, such that if $\omega \geq C\rho$ and*

$$|\{(x,t) \in Q : u(x,t) > \mu_+ - \omega\}| \leq s_1 |Q|,$$
$$|\{(x,t) \in Q : u(x,t) > \mu_+ - \omega\}|_\nu \leq s_1^\nu |Q|_\nu,$$

then we have

$$\operatorname*{ess\,sup}_{\frac{1}{2}Q} u(x,t) \leq \mu_+ - a\omega.$$

Lemma 4.2.11. *Let u be a supersolution to (4.2.1) in the cylinder $Q = K_{2\rho}^y \times [t_1 - \theta h(y,2\rho), t_1]$. Let $\mu_- \leq \operatorname{ess\,inf}_Q u(x,t)$. Then for any $\omega > 0$ and $a \in (0,1)$ there exist numbers s_1^ν and s_1, which depend only on the data, a, and $\theta\omega^{p-2}$,* *such that if* $\omega \geq C\rho$ *and*

$$|\{(x,t) \in Q \,:\, u(x,t) < \mu_- + \omega\}| \leq s_1|Q|,$$
$$|\{(x,t) \in Q \,:\, u(x,t) < \mu_- + \omega\}|_\nu \leq s_1^\nu|Q|_\nu,$$

then we have

$$\operatorname*{ess\,inf}_{\frac{1}{2}Q} u(x,t) \geq \mu_- + a\omega.$$

Remark. The constants s_1 and s_1^ν in Lemmas 4.2.10, 4.2.11 are the same for the same values of ω, a and can be taken as

$$s_1^\nu = \frac{\gamma}{\theta\omega^{p-2}}\left[(1-a)^p\left(1+\left(\theta\omega^{p-2}\right)^{-1/h}\right)\right]^{\frac{h}{1-h}}, \quad s_1 = \theta\omega^{p-2}s_1^\nu.$$

Here $\gamma = \gamma(data)$ and $h > 1$ is a constant determined by ν. The constant h comes from corollary 4.2.9. We prove only Lemma 4.2.11, the proof of Lemma 4.2.10 being completely the same.

Proof of Lemma 4.2.11. For $j = 0, 1, 2, \ldots$ denote

$$\rho_j = \rho(1+2^{-j}), \quad \omega_j = \omega\left(a + (1-a)2^{-j}\right),$$
$$h_j = h(y, 2\rho) + \frac{h(y,2\rho) - h(y,\rho)}{2^j}, \quad k_j = \mu_- + \omega_j,$$
$$K_j = K^y_{\rho_j}, \quad Q_j = K_j \times [t_1 - \theta h_j, t_1].$$

Thus, $K_\infty = K^y_\rho$ and $Q_\infty = \frac{1}{2}Q$. We introduce a sequence of piecewise-smooth cut-off functions $\phi_j(x)$ and $\psi_j(t)$ such that:

1. $0 \leq \phi_j(x) \leq 1$, $\phi_j(x) = 1$ on K_{j+1}, $\phi_j(x) = 0$ outside K_j and $|D\phi_j| \leq 2(\rho_j - \rho_{j+1})^{-1}$;
2. $\psi_j(t) = 1$ for $t \geq t_1 - \theta h_{j+1}$, $\psi_j(t) = 0$ for $t \leq t_1 - \theta h_j$, and $0 \leq (\psi_j)_t \leq 2\left[\theta(h_j - h_{j+1})\right]^{-1}$.

It is clear that the functions $\xi_j(x,t) = \phi_j(x)\psi_j(t)$ are piecewise-smooth cut-off functions such that $0 \leq \xi_j \leq 1$ on Q_j, $\xi_j(x,t) = 1$ on Q_{j+1}, $\xi_j = 0$ on the parabolic boundary of Q_j, $|D\xi_j| \leq 2[\rho_j - \rho_{j+1}]^{-1}$, $|(\xi_j)_t| \leq 2\left[\theta(h_j - h_{j+1})\right]^{-1}$.

Writing energy estimate (4.2.17) over the cylinder Q_j with $k = k_j$ and $\xi = \xi_j$ we see that

$$\max_{t_1 - \theta h_j \leq t \leq t_1} \int_{K_j} (u-k_j)_-^2 \xi_j^p dx + \iint |D\left[(u-k_j)_-\xi_j\right]|^p \nu dxdt$$
$$\leq \gamma\left(\iint_{Q_j} \frac{(u-k_j)_-^2}{\theta(h_j - h_{j+1})} dxdt + \iint_{Q_j} \frac{(u-k_j)_-^p}{(\rho_j - \rho_{j+1})^p}\nu dxdt\right.$$
$$\left. + \iint_{Q_j} (u-k_j)_-^p \nu dxdt + C^p \iint_{Q_j} \chi_{\{u<k_j\}}\nu dxdt\right). \tag{4.2.18}$$

Here $\gamma = \gamma(n, p, C_0, C_1, C_2)$. Note, that from (4.2.8) it follows that

$$h_j - h_{j+1} \geq c2^{-j}h(y, 2\rho) \geq c2^{-j}h(y, \rho).$$

For the family of sets

$$A_j = \{(x,t) \in Q_j : u(x,t) < k_j\}$$

we denote

$$Y_j = \frac{|A_j|}{|Q|}, \quad X_j = \frac{\nu(A_j)}{\nu(Q)}.$$

From (4.2.18) we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t_1-\theta h_j \leq t \leq t_1} \int_{K_j} (u-k_j)_-^2 \xi_j^p dx + C_0 \iint |D[(u-k_j)_- \xi_j]|^p \nu dxdt \\
&\leq \gamma \frac{\omega^p 2^{pj}}{\rho^p} \left(\frac{|Q|\rho^p}{\theta \omega^{p-2} h(y, 2\rho)} Y_j + \nu(Q) X_j (1 + \rho^p + (\omega^{-1} C\rho)^p) \right) \\
&\qquad \leq \gamma \frac{\omega^p 2^{pj}}{\rho^p} \nu(Q_j) \left(\frac{Y_j}{\theta \omega^{p-2}} + X_j \right). \qquad (4.2.19)
\end{aligned}$$

In the last passage we used that 1) $\rho_j \leq diam\Omega$, 2) $\omega > C\rho$, and 3) $|Q|\rho^p \leq \nu(Q)h(y, 2\rho)$.

Let $h \in (1, h_0)$, where h_0 is a number contained in the statement of Corollary 4.2.9.

Since for $(x,t) \in A_{j+1}$ we have

$$(u-k_j)_-(x,t) \geq k_j - k_{j+1} = \frac{1-a}{2^{j+1}}\omega$$

we can estimate $|A_{j+1}|$ as

$$\begin{aligned}
&\left(\frac{1-a}{2^{j+1}\omega}\right)^p |A_{j+1}| \leq \iint_{A_{j+1}} (u-k_j)_-^p dxdt \\
&\leq \iint_{A_j} (u-k_j)_-^p \xi_j^p dxdt \leq \left(\iint_{Q_j} (u-k_j)_-^{ph} \xi_j^{ph} dxdt \right)^{1/h} |A_j|^{1-1/h} \\
&\leq \gamma |A_j|^{1-1/h} \left(|K_j| \left(\frac{1}{|K_j|} \operatorname*{ess\,sup}_{t_1-\theta h_j \leq t \leq t_1} \int_{K_j} (u-k_j)_-^p \xi_j^p dx \right)^{h-1} \right. \\
&\qquad \left. \times \frac{\rho_j^p}{|K_j|_\nu} \iint_{Q_j} |D[(u-k_j)_- \xi_j]|^p \nu(x) dxdt \right)^{1/h} \\
&\leq \gamma \left(\frac{\rho^p |K_\infty|^{2-h}}{\nu(K_\infty)} \right)^{1/h} \frac{\omega^p 2^{pj}}{\rho^p} \nu(Q) \left(X_j + \frac{Y_j}{\theta \omega^{p-2}} \right) \omega^{(p-2)(1-1/h)} |A_j|^{1-1/h}.
\end{aligned}$$

Dividing both sides by $|Q|_{j+1}$, we arrive at

$$Y_{j+1} \leq \frac{\gamma 4^{pj}}{(1-a)^p} \omega^{(p-2)(1-1/h)} M_\rho \left(X_j + \frac{Y_j}{\theta \omega^{p-2}} \right) Y_j^{1-1/h}, \tag{4.2.20}$$

where we estimate

$$M_\rho = \left(\frac{\rho^p |K_\infty|^{2-h}}{\nu(K_\infty)} \right)^{1/h} |Q|^{-1/h} \frac{\nu(Q)}{\rho^p} \leq c \left(\frac{\rho^p |K_\infty|^{2-h}}{\nu(K_\infty)} \right)^{1/h} |Q|^{1-1/h} \frac{1}{h(y,\rho)}$$
$$\leq c\theta^{1-1/h} \left(\frac{\rho^p |K_\rho^y|}{\nu(K_\rho^y) h(y,\rho)} \right)^{1/h} \leq c\theta^{1-1/h}.$$

Thus, we have

$$Y_{j+1} \leq \frac{\gamma 4^{pj}}{(1-a)^p} (\theta \omega^{p-2})^{1-1/h} \left(X_j + \frac{Y_j}{\theta \omega^{p-2}} \right) Y_j^{1-1/h}. \tag{4.2.21}$$

Arguing in the similar manner for X_j we obtain

$$X_{j+1} \leq \frac{\gamma 4^{pj}}{(1-a)^p} (\theta \omega^{p-2})^{1-1/h} \left(X_j + \frac{Y_j}{\theta \omega^{p-2}} \right) X_j^{1-1/h}. \tag{4.2.22}$$

Here instead of M_ρ we have

$$N_\rho = \left(\rho^p |K_\rho^y|^{1-h}\right)^{1/h} \rho^p \nu(Q)^{1-1/h}$$
$$= \left(\rho^{p(1-h)} |K_\rho^y|^{1-h} \nu(K_\rho^y)^{h-1} h(y,\rho)^{h-1} \right) \theta^{1-1/h}.$$

We use the relation (4.2.6) to conclude that $N_\rho \leq c\theta^{1-1/h}$. Denote

$$M_j = X_j + \frac{Y_j}{\theta \omega^{p-2}}.$$

From (4.2.21), (4.2.22) we derive

$$M_{j+1} \leq \frac{\gamma 4^{pj}}{(1-a)^p} (\theta \omega^{p-2})^{1-1/h} M_j \left(X_j^{1-1/h} + \frac{1}{\theta \omega^{p-2}} Y_j^{1-1/h} \right)$$
$$\leq \frac{\gamma 4^{pj}}{(1-a)^p} (\theta \omega^{p-2})^{1-1/h} \left(1 + \left(\theta \omega^{p-2}\right)^{-1/h} \right) M_j^{1+(1-1/h)}. \tag{4.2.23}$$

Denote $\alpha = 1 - 1/h$. The well-known hypergeometric convergence lemma ([81], [80]) implies that $M_j \to 0$ as $j \to \infty$ provided that

$$M_0 \leq (4^p)^{-1/\alpha^2} (1-a)^{p/\alpha} \frac{1}{\theta \omega^{p-2}} \left(1 + \left(\theta \omega^{p-2}\right)^{-1/h} \right)^{\frac{h}{1-h}} \gamma^{-1/\alpha}. \tag{4.2.24}$$

It is clear that the condition (4.2.24) is satisfied if

$$X_0 < s_1^\nu := \gamma (1-a)^{p/\alpha} \frac{1}{\theta \omega^{p-2}} \left(1 + \left(\theta \omega^{p-2}\right)^{-1/h} \right)^{\frac{h}{1-h}}, \tag{4.2.25}$$

$$Y_0 < s_1 := \gamma (1-a)^{p/\alpha} \left(1 + \left(\theta \omega^{p-2}\right)^{-1/h} \right)^{\frac{h}{1-h}}, \tag{4.2.26}$$

with sufficiently small constant γ (depending on the data). $\square$

Lower bounds for nonnegative supersolutions. First, we state the following lemma. Its proof is in fact a simplified version of the proof of Lemmas 4.2.10, 4.2.11.

Lemma 4.2.12. *Let u be a nonnegative supersolution of* (4.2.1) *in the cylinder $Q = K^y_{2\rho} \times [t_1 - \theta h(y, 2\rho), t_1]$. Suppose that* $\operatorname{ess\,inf}_{K^y_{2\rho}} u(x, t_1 - \theta h(y, 2\rho)) \geq k$*, and $k \geq C\rho$. There exists $\delta = \delta(data)$ such that if $\theta < \delta k^{2-p}$ then* $\operatorname{ess\,inf}_{K^y_\rho} u(x, t_1) \geq k/2$*. The constant δ is independent of both u and k.*

Proof. Let ρ_j and $\phi_j(x)$ be the same as in the proof of the previous lemma. Consider the sequence of levels $k_j = \frac{k}{2}\left(1 + 2^{-j}\right)$ and the sequence of cylinders $Q_j = K^y_{\rho_j} \times [t_1 - \theta h(y, 2\rho), t_1]$. Denote

$$A_j = \{(x,t) \in Q_j : u(x,t) \leq k_j\}, \quad X_j = \frac{\nu(A_j)}{\nu(Q_j)} \quad Y_j = \frac{|A_j|}{|Q_j|}.$$

Write for u energy estimate (4.2.17) with $k = k_j$ and $\xi(x,t) = \phi_j(x)$ over the cylinder Q_j. The integral over the lower base of the cylinder Q_j disappears since for each k_j we have $(u - k_j)_-(x, t_1 - \theta h(y, 2\rho)) = 0$ a.e. in $K^y_{2\rho}$. Then we repeat the calculations in the proof of Lemma 4.2.11. Moreover, they are simplified since the term containing $(\xi_j)_t$ disappears. Instead of (4.2.21) and (4.2.22) we obtain

$$Y_{j+1} \leq \gamma 4^{pj} X_j \left(\theta k^{p-2}\right)^{1-1/h} Y_j^{1-1/h},$$
$$X_{j+1} \leq \gamma 4^{pj} \left(\theta k^{p-2}\right)^{1-1/h} X_j^{1+(1-1/h)}.$$

We use again the Ladyzhenskaja-Uraltseva lemma to conclude that the sufficient condition for X_∞ to be 0 is

$$X_0 \leq \frac{\delta}{\theta k^{p-2}},$$

where $\delta = \delta(data)$. Since $X_0 \leq 1$ the last condition is clearly satisfied if $\theta \leq \delta k^{2-p}$. $\square$

This lemma almost immediately implies the following corollary.

Corollary 4.2.13. *Let u be a nonnegative supersolution to* (4.2.1) *in the cylinder $K^y_{2\rho} \times [t_1, t_1 + T]$. Let* $\operatorname{ess\,inf}_{K^y_{2\rho}} u(x, t_1) \geq k$*. Let $k \geq C\rho$. Then for all $t \leq t_1 + \delta (C\rho)^{2-p} h(y, 2\rho)$ we have*

$$\operatorname*{ess\,inf}_{K^y_\rho} u(x,t) \geq \frac{k}{2}\left(1 + \frac{t - t_1}{\delta k^{2-p} h(y, 2\rho)}\right)^{\frac{1}{2-p}}. \tag{4.2.27}$$

Here δ is the constant from the statement of Lemma 4.2.12. If $C = 0$ the estimate (4.2.27) *is valid for all $t \in [t_1, T]$.*

Proof. It is clear, that for all $\tau \in [0, 1]$ we have

$$\operatorname*{ess\,inf}_{K^y_{2\rho}} u(x, t_1) \geq \tau k. \tag{4.2.28}$$

If $t - t_1 \leq \delta k^{2-p} h(y, 2\rho)$, then Lemma 4.2.12 yields $u(x,t) \geq k/2$ a.e. in K^y_ρ. Now assume that $t - t_1 > \delta k^{2-p} h(y, 2\rho)$. Take in (4.2.28)

$$\tau = \left(\frac{\delta k^{2-p} h(y, 2\rho)}{t - t_1}\right)^{\frac{1}{p-2}}$$

and apply Lemma 4.2.12 with k replaced by τk. This gives

$$u(x,t) \geq \left(\frac{\delta k^{2-p} h(y, 2\rho)}{t - t_1}\right)^{\frac{1}{p-2}} \frac{k}{2} \quad \text{a.e. in} \quad K^y_\rho.$$

provided that

$$\tau k = \left(\frac{\delta h(y, 2\rho)}{t - t_1}\right)^{\frac{1}{p-2}} \geq C\rho \Leftrightarrow t - t_1 \leq \delta (C\rho)^{2-p} h(y, 2\rho).$$

Combination of the estimates for $t \leq t_1 + \delta k^{2-p} h(y, 2\rho)$ and $t > t_1 + \delta k^{2-p} h(y, 2\rho)$ concludes the proof. $\square$

Expansion of positivity. This argument was the key step in the approach of [28]. We reproduce it here in the weighted case.

Lemma 4.2.14. *Let v be a nonnegative supersolution of* (4.2.1) *in the cylinder $Q = K^y_{8\rho} \times [0, T]$. Let for all $t \in [0, T]$*

$$|\{x \in K^y_{4\rho} : v(x,t) \geq 1\}| \geq \kappa |K^y_{4\rho}|,$$

where $\kappa = const > 0$. Then for any $\varepsilon > 0$ there exist positive numbers σ_ε and θ_ε such that if the cylinder $Q_\varepsilon = K^y_{4\rho} \times [\theta_\varepsilon h(y, 4\rho), 2\theta_\varepsilon h(y, 4\rho)] \subset Q$ and $\sigma_\varepsilon \geq C\rho$, then

$$|\{(x,t) \in Q_\varepsilon : v(x,t) < \sigma_\varepsilon\}| \leq \varepsilon |Q_\varepsilon|.$$

Moreover, σ_ε and θ_ε depend only on ε, κ and the data and satisfy $\sigma_\varepsilon^{p-2} \theta_\varepsilon = 1$. The value of θ_ε increases and the value of σ_ε decreases as ε decreases.

Proof. Denote $k_j = 2^{-j}$ for $j = 0, 1, \ldots, j_*$, where j_* will be chosen later. Denote $Q_1 = K^y_{4\rho} \times [\theta h(y, 4\rho), 2\theta h(y, 4\rho)]$ and $Q_2 = K^y_{8\rho} \times [0, 2\theta h(y, 4\rho)]$, where the constant θ will be specified later. Take the piecewise-smooth cut-off function $\xi(x,t)$ such that $\xi = 1$ on Q_1, $0 \le \xi \le 1$ on Q_2, ξ vanishes on the parabolic boundary of Q_2, $|\xi_t| \le \frac{2}{\theta h(y,4\rho)}$ and $|D\xi| \le \frac{2}{4\rho}$.

Using energy estimate (4.2.17) for $k = k_j$ and $\xi(x,t)$ we obtain

$$\max_{\theta h(y,4\rho)\le\tau\le 2\theta h(y,4\rho)} \int_{K^y_{4\rho}} (v-k_j)_-^2\,dx + \iint_{Q_1} |D(v-k_j)_-|^p \nu\,dxdt$$

$$\le \gamma\left(\frac{|Q_2|k_j^2}{\theta h(y,4\rho)} + \nu(Q_2)\left(\frac{k_j^p}{(4\rho)^p} + k_j^p + C^p\right)\right)$$

$$\le \frac{\gamma k_j^p}{(4\rho)^p}\nu(Q_2)\left(1+\rho^p + k_j^{-p}(4C\rho)^p + \frac{|Q_2|(4\rho)^p}{\nu(Q_2)h(y,4\rho)}\cdot\frac{1}{\theta k_j^{p-2}}\right).$$

Take $\theta = k_{j_*}^{2-p}$ and assume that $k_{j^*} \ge C\rho$. Using (4.2.7) and $\rho \le diam\Omega$ we obtain

$$\iint_{Q_1} |D(v-k_j)_-|^p \nu\,dxdt \le \frac{\gamma k_j^p}{(4\rho)^p}\nu(Q_1), \quad \text{for all} \quad j = 1,\dots,j_*$$

with $\gamma = \gamma(data)$. Denote

$$A_j = \{(x,t)\in Q_1 : v(x,t) < k_j\} \quad \text{and} \quad A_j(\tau) = \{x \in K^y_{4\rho} : v(x,\tau) < k_j\}.$$

The De Giorgi-Poincaré inequality (see [25], [81], [80]) implies

$$(k_j - k_{j+1})|A_{j+1}(\tau)| \le \frac{\gamma\rho^{n+1}}{|K^y_{4\rho}\setminus A_j(\tau)|}\int_{A_j(\tau)\setminus A_{j+1}(\tau)} |D(v-k_j)_-|\,dx,$$

where $\gamma = \gamma(n)$. We use the condition of the lemma and integrate the last inequality over $\tau \in [\theta h(y,4\rho), 2\theta h(y,4\rho)]$. We have

$$\frac{k_j}{2}|A_{j+1}| \le \frac{\gamma\rho}{\kappa}\iint_{A_j\setminus A_{j+1}} |D(v-k_j)_-|\,dxdt$$

$$\le \frac{\gamma\rho}{\kappa}\left(\iint_{A_j\setminus A_{j+1}} |D(v-k_j)_-|^p\nu\,dxdt\right)^{\frac{1}{p}}\left(\iint_{A_j\setminus A_{j+1}} \nu^{\frac{1}{1-p}}\,dxdt\right)^{\frac{p-1}{p}}$$

$$\le \frac{\gamma}{\kappa}k_j\left(\nu(Q_1)\right)^{1/p}\left(\iint_{A_j\setminus A_{j+1}} \nu^{\frac{1}{1-p}}\,dxdt\right)^{\frac{p-1}{p}}.$$

Now cancel k_j on both sides of the last inequality and raise it to the power $\frac{p}{p-1}$. This yields

$$|A_{j+1}|^{\frac{p}{p-1}} \le \gamma\left(\nu(Q_1)\right)^{\frac{1}{p-1}}\nu^{\frac{1}{1-p}}(A_j\setminus A_{j+1}).$$

Summation of the last inequality over $j = 0, 1, \dots, j_* - 1$ yields

$$j_* |A_{j_*}|^{\frac{p}{p-1}} \le \sum_{j=0}^{j_*-1} |A_j|^{\frac{p}{p-1}} \le \gamma \nu(Q_1)^{\frac{1}{p-1}} \nu^{\frac{1}{1-p}}(A_0 \setminus A_{j_*}) \le \gamma |Q_1|_\nu^{\frac{1}{p-1}} \nu^{\frac{1}{1-p}}(Q_1).$$

Since $\nu \in A_{1+\frac{p}{n}} \subset A_p$, the last inequality implies

$$j_* |A_{j_*}|^{\frac{p}{p-1}} \le \gamma |Q_1|^{\frac{p}{p-1}}.$$

We have proved the lemma with

$$\sigma_\varepsilon = 2^{-j_*}, \quad \theta_\varepsilon = \sigma_\varepsilon^{2-p}, \quad \varepsilon = \left(\frac{\gamma}{j_*}\right)^{\frac{p-1}{p}}.$$

□

We use the result of Lemma 4.2.14 to prove

Corollary 4.2.15. *Let the conditions of Lemma 4.2.14 be satisfied. Denote*

$$Q(\theta) = K_{2\rho}^y \times [2\theta h(y, 4\rho) - \theta h(y, 2\rho), 2\theta h(y, 4\rho)].$$

There exists $\theta_0 = \theta_0(data, \kappa)$ such that for all $\theta \ge \theta_0$ we have

$$\operatorname*{ess\,inf}_{Q(\theta)} v(x, t) \ge \frac{1}{2} \theta^{\frac{1}{2-p}}$$

provided that $\theta^{\frac{1}{2-p}} \ge 4C\rho$ and $T \ge 2\theta h(y, 4\rho)$.

Proof. Let $\varepsilon > 0$ and $\sigma_\varepsilon, \theta_\varepsilon$ be as in the statement of Lemma 4.2.14. Consider the family of the cylinders $Q_\varepsilon = K_{4\rho}^y \times [\theta_\varepsilon h(y, 4\rho), 2\theta_\varepsilon h(y, 4\rho)]$. We use Lemma 4.2.11 for v in the cylinder Q_ε with $\mu_- = 0$, $\omega = \sigma_\varepsilon$ and $a = \frac{1}{2}$ to find corresponding numbers $s_1(\varepsilon)$ and $s_1^\nu(\varepsilon)$. Since $\theta_\varepsilon \sigma_\varepsilon^{p-2} = 1$, they can be chosen independently of ε. Thus we can drop ε and refer to them as s_1 and s_1^ν.

Denote

$$m(\tau) = |\{x \in K_{4\rho}^y : v(x, t) < \sigma_\varepsilon\}|, \quad m_\nu(\tau) = \nu(\{x \in K_{4\rho}^y : v(x, t) < \sigma_\varepsilon\}),$$
$$M(\varepsilon) = |\{(x, t) \in Q_\varepsilon : v(x, t) < \sigma_\varepsilon\}|, \quad M_\nu(\varepsilon) = \nu(\{(x, t) \in Q_\varepsilon : v(x, t) < \sigma_\varepsilon\}).$$

Lemma 4.2.14 yields $M(\varepsilon) \le \varepsilon |Q_\varepsilon|$. Using property (4.2.9) of the Muckenhoupt weights we estimate

$$\begin{aligned} M_\nu(\varepsilon) &= \int_{\theta h(y,4\rho)}^{2\theta h(y,4\rho)} m_\nu(\tau) d\tau \le c \int_{\theta h(y,4\rho)}^{2\theta h(y,4\rho)} \nu(K_{4\rho}^y) \left(\frac{m(\tau)}{|K_{4\rho}^y|}\right)^\kappa \\ &\le c\nu(K_{4\rho}^y)(\theta h(y, 4\rho))^{1-\kappa} \left(\int_{\theta h(y,4\rho)}^{2\theta h(y,4\rho)} \frac{m(\tau)}{|K_{4\rho}^y|}\right)^\kappa = c\nu(Q_\varepsilon) \left(\frac{M(\varepsilon)}{|Q_\varepsilon|}\right)^\kappa \\ &\le c\nu(Q_\varepsilon)\varepsilon^\kappa. \end{aligned}$$

Therefore, for

$$\varepsilon \le \varepsilon_0 = \min\left(s_1, \left(\frac{s_1^\nu}{c}\right)^{\frac{1}{\kappa}}\right)$$

we have

$$|\{(x,t) \in Q_\varepsilon : v(x,t) < \sigma_\varepsilon\}| \le s_1 |Q_\varepsilon| \quad \text{and}$$
$$\nu(\{(x,t) \in Q_\varepsilon : v(x,t) < \sigma_\varepsilon\}) \le s_1^\nu \nu(Q_\varepsilon).$$

Take $\theta_0 = \theta_{\varepsilon_0}$. We apply Lemma 4.2.11 in Q_ε to complete the proof. □

Now we have all technical ingredients to prove the core ingredient of the proof. In [28] it was dubbed the 'expansion of positivity'.

Lemma 4.2.16. *Let u be a nonnegative supersolution of* (4.2.1) *in the cylinder $K_{8\rho}^y \times [0,T]$. Let $\operatorname{ess\,inf}_{x\in K_\rho^y} u(x,0) \ge k$. There exist such constants $\gamma_1 = \gamma_1(data)$, $\gamma_2 = \gamma_2(data)$ and $\gamma_3 = \gamma_3(data)$ that*

$$\operatorname*{ess\,inf}_{x\in K_{2\rho}^y} u(x, \gamma_1 k^{2-p} h(y,\rho)) \ge \gamma_2 k, \tag{4.2.29}$$

provided that $T \ge \gamma_1 k^{2-p} h(y,\rho)$ and $k \ge \gamma_3 C\rho$.

Proof. We can assume from the beginning that $\gamma_3 \ge 1$, i.e. $k \ge C\rho$. Denote

$$\psi(t) = \frac{k}{2}\left(1 + \frac{t}{\delta k^{2-p} h(y,\rho)}\right)^{\frac{1}{2-p}},$$

where δ is a constant from the statement of Lemma 4.2.12. From Corollary 4.2.13 we have

$$\operatorname*{ess\,inf}_{x\in K_{\rho/2}^y} u(x,t) \ge \psi(t)$$

for $t \le \delta (C\rho)^{2-p} h(y,\rho)$. We change in equation (4.2.1) the variables as

$$u(x,t) = v(x,t)\psi(t),$$
$$d\tau = \psi^{p-2}(t)dt, \quad \tau(0) = 0.$$

It is easy to see that

$$\tau = \frac{\delta}{2^{p-2}} h(y,\rho) \ln\left(1 + \frac{t}{\delta k^{2-p} h(y,\rho)}\right),$$
$$\psi(t(\tau)) = \frac{k}{2}\exp\left[\frac{2^{p-2}\tau}{(2-p)\delta h(y,\rho)}\right],$$

and for all

$$\tau \le \frac{\delta}{2^{p-2}} h(y,\rho) \ln\left(1 + \left(\frac{k}{C\rho}\right)^{p-2}\right)$$

we have

$$\operatorname*{ess\,inf}_{K^y_{\rho/2}} v(x,\tau) \geq 1.$$

Let us show that $v(x,\tau)$ is a supersolution to the equation similar to (4.2.1). Arguing formally, one has

$$\begin{aligned}\frac{\partial}{\partial\tau}v(x,\tau) &= \frac{\partial}{\partial t}v(x,t)\cdot\frac{dt}{d\tau} = \psi^{2-p}(t)\left[\frac{u_t}{\psi} - \frac{\psi'}{\psi^2}u\right]\\ &\geq \psi^{1-p}(t)u_t = \psi^{1-p}(t)\operatorname{div}\mathbf{A}(x,t,u,Du) + \psi^{1-p}(t)B(x,t,u,Du)\\ &= \operatorname{div}\mathbf{A}_1(x,\tau,v,Dv) + B_1(x,\tau,v,Dv),\end{aligned}$$

where

$$\begin{aligned}\mathbf{A}_1(x,\tau,v,Dv) &= \psi^{1-p}\mathbf{A}(x,t,v\psi,\psi Dv),\\ B_1(x,\tau,v,Dv) &= \psi^{1-p}B(x,t,v\psi,\psi Dv).\end{aligned}$$

It is obvious that

$$\begin{aligned}\mathbf{A}_1(x,\tau,v,Dv)\cdot Dv &= \psi^{-p}\mathbf{A}(x,t,u,Du)\cdot Du\\ &\geq \nu(x)C_0\psi^{-p}|Du|^{-p} - \nu(x)\psi^{-p}C^p = C_0\nu(x)|Dv|^p - \nu(x)\psi^{-p}C^p.\end{aligned}$$

Similarly

$$\begin{aligned}|\mathbf{A}_1(x,\tau,v,Dv)| &\leq C_1\psi^{1-p}|\psi Dv|^{p-1}\nu(x) + \nu(x)C^{p-1}\psi^{1-p}\\ &= C_1|Dv|^{p-1}\nu(x) + \nu(x)C^{p-1}\psi^{1-p}\end{aligned}$$

and

$$|B_1(x,\tau,v,Dv)| \leq C_2\nu(x)|Dv|^{p-1} + \nu(x)C^{p-1}\psi^{1-p}.$$

The formal verification can be made by changing the variables in the definition of a weak solution.

By Corollary 4.2.15, for any $\theta \geq \theta_0(data)$ we have

$$\operatorname*{ess\,inf}_{K^y_{2\rho}} v(x, 2\theta h(y,4\rho)) \geq \frac{1}{2}\theta^{\frac{1}{2-p}}$$

provided that

$$\theta^{\frac{1}{2-p}} \geq 4 \sup_{0\leq\tau\leq 2\theta h(y,4\rho)} \frac{C\rho}{\psi(t(\tau))}. \tag{4.2.30}$$

and

$$2\theta h(y,4\rho) \leq \frac{\delta}{2^{p-2}}h(y,\rho)\ln\left(1+\left(\frac{k}{C\rho}\right)^{p-2}\right). \tag{4.2.31}$$

Denote $\mathcal{S} = \sup\limits_{y,\rho}\frac{h(y,4\rho)}{h(y,\rho)}$. Let $\theta = \theta_0\frac{h(y,\rho)}{h(y,4\rho)}\mathcal{S}$. It is clear that $\theta \in [\theta_0, \theta_0\mathcal{S})$.

Inequalities (4.2.30) and (4.2.31) clearly hold if k satisfies both

$$k \geq 8\theta^{\frac{1}{p-2}} C\rho \exp\left[\frac{2^{p-2}h(y,4\rho)2\theta}{(p-2)\delta h(y,\rho)}\right] = 8\theta^{\frac{1}{p-2}} C\rho \exp\left[\frac{2^{p-1}\theta_0}{(p-2)\delta}\mathcal{S}\right]$$

and

$$k \geq \left(\exp\left[\frac{2\theta h(y,4\rho)2^{p-2}}{\delta h(y,\rho)}\right] - 1\right)^{\frac{1}{p-2}} C\rho = \left(\exp\left[\theta_0 2^{p-1}\delta^{-1}\mathcal{S}\right] - 1\right)^{\frac{1}{p-2}} C\rho.$$

Thanks to our choice of θ, the last two inequalities are satisfied if $k \geq \gamma_3 C\rho$ with the constant $\gamma_3 = \gamma_3(data)$.

Returning to the original variables, we obtain

$$\operatorname*{ess\,inf}_{K^y_{2\rho}} u(x,t_*) \geq \frac{1}{2}\theta^{\frac{1}{2-p}}\psi(t_*)$$

$$= \frac{k}{4}\theta^{\frac{1}{2-p}} \exp\left[\frac{1}{2-p}\cdot\frac{\theta}{2^{1-p}\delta}\cdot\frac{h(y,4\rho)}{h(y,\rho)}\right] \geq \frac{k}{4}\left(\theta_0\mathcal{S}\right)^{\frac{1}{2-p}} \exp\left[\frac{\theta_0 2^{p-1}\mathcal{S}}{(2-p)\delta}\right],$$

where

$$t_* = \delta k^{2-p}h(y,\rho)\left[\exp\left(\frac{\theta 2^{p-1}h(y,4\rho)}{\delta h(y,\rho)}\right) - 1\right]$$
$$= \delta k^{2-p}h(y,\rho)\left[\exp\left(2^{p-1}\delta^{-1}\theta_0\mathcal{S}\right) - 1\right].$$

Thus we have proved the assertion of the lemma with

$$\gamma_1 = \delta\left[\exp\left(2^{p-1}\delta^{-1}\theta_0\mathcal{S}\right) - 1\right], \quad \gamma_2 = \frac{1}{4}\left(\theta_0\mathcal{S}\right)^{\frac{1}{2-p}} \exp\left[\frac{2^{p-1}\theta_0}{\delta(2-p)}\mathcal{S}\right].$$

Note that $\gamma_2 \to 0$ and $\gamma_3 \to +\infty$ as $p \to 2+$. □

Remark. In fact, we have proved that $\operatorname{ess\,inf}_{x\in K^y_{2\rho}} u(x,t) \geq \gamma_2 k$ for all

$$\gamma_1' k^{2-p}h(y,\rho) \leq t \leq \gamma_1 k^{2-p}h(y,\rho),$$

where

$$\gamma_1' = \delta\left[\exp\left(2^{p-2}\delta^{-1}\theta_0\mathcal{S}\right) - 1\right].$$

Proof of Theorem 4.2.1. The idea of the first part of the proof (concentration of positivity) is essentially due to Krylov and Safonov ([77]). Consider the family of expanding cylinders

$$Q_\tau = K^{x_0}_{\tau\rho} \times \left[t_0 - h(\tau)k^{2-p}, t_0\right],$$

where

$$h(\tau) = \theta(1-(1-\tau)^\kappa)h(x_0,\rho), \quad \kappa = const > 0,$$

and τ ranges over $(0,1]$. Let us show that κ can be chosen such that for any point $(x_\tau, t_\tau) \in Q_\tau$ and some constant $\theta_1 > 0$

$$K^{x_\tau}_{\frac{1-\tau}{2}\rho} \times \left[t_\tau - \theta_1 h(x_\tau, \frac{1-\tau}{2}\rho)k^{2-p}, t_\tau\right] \subset Q_{\frac{1+\tau}{2}}.$$

Since for all $\tau \in [0,1]$ we obviously have

$$\tau^p + \left(\frac{1-\tau}{2}\right)^p \leq \left(\frac{1+\tau}{2}\right)^p,$$

we only need to check that

$$\theta(1-(1-\tau)^\kappa)h(x_0,\rho) + \theta_1 h(x_\tau, \frac{1-\tau}{2}\rho) \leq \theta h(x_0,\rho)\left(1-\left(1-\frac{1+\tau}{2}\right)^\kappa\right).$$

The last inequality is clearly satisfied if

$$\theta_1 \leq \theta \inf \frac{h(x_0,\rho)(1-\tau)^\kappa}{h(x_\tau, \frac{1-\tau}{2}\rho)}(1-2^{-\kappa}), \tag{4.2.32}$$

where the infimum is taken over all $\tau \in [0,1]$ and $x_\tau \in K^{x_0}_{\tau\rho}$.

Now we use $\nu^{-n/p} \in A_{1+n/p}$ and property (4.2.9) of the Muckenhoupt weights to deduce that there exist positive constants $\delta \in (0,1]$ and c such that

$$\frac{h(x_\tau, \frac{1-\tau}{2}\rho)}{h(x_0,\rho)} \leq c\left(\frac{1-\tau}{2}\right)^{\delta p}$$

for all $\tau \in [0,1]$ and $x_\tau \in K^{x_0}_{\tau\rho}$. Take $\kappa = \delta p$. Then inequality (4.2.32) is satisfied if we take

$$\theta_1 = \theta\frac{2^\kappa - 1}{c}.$$

On the interval $(0,1]$ define the functions

$$a(\tau) = \operatorname*{ess\,sup}_{Q_\tau} u(x,t), \quad b(\tau) = (1-\tau)^{-\beta}k,$$

where $\beta > 0$ is a constant which will be specified later. Note that $b(\tau)$ grows to infinity as $\tau \to 1-0$ while $a(\tau)$ stays uniformly bounded. Set

$$\tau_0 = \inf\{\tau \in [0,1) : a(\xi) \leq b(\xi) \quad \text{for all} \quad \xi > \tau\}.$$

Denote

$$k_1 = (1-\tau_0)^{-\beta}k \quad \text{and} \quad R = \frac{1-\tau_0}{2}\rho.$$

Clearly,

$$k_1 \geq k \quad \text{and} \quad k_1 \geq CR.$$

One can easily see that there exists a cylinder

$$Q_2 = K_R^{x_1} \times [t_1 - \theta_1 h(x_0, R) k_1^{2-p}, t_1],$$

where $(x_1, t_1) \in Q_{\tau_0}$, such that

$$\operatorname*{ess\,sup}_{\frac{1}{2}Q_2} u(x,t) \geq k_1.$$

Note that by our choice of parameters $Q_2 \subset Q_{\frac{1+\tau_0}{2}}$. Consequently,

$$\operatorname*{ess\,sup}_{Q_2} u(x,t) < \left(1 - \frac{1+\tau_0}{2}\right)^{-\beta} k = 2^\beta (1-\tau_0)^{-\beta} k = 2^\beta k_1.$$

The next step demonstrates that in Q_2 the measure of the set where $u(x,t) \geq \frac{k_1}{2}$ is relatively big.

Lemma 4.2.17. *There exist positive numbers $\xi_1 = \xi_1(data, \beta, \theta)$ and $\xi_2 = \xi_2(data, \beta, \theta)$ such that*

$$|\{(x,t) \in Q_2 : u(x,t) > k_1/2\}| > \xi_1 |Q_2| \tag{4.2.33}$$

and

$$\nu(\{(x,t) \in Q_2 : u(x,t) > k_1/2\}) > \xi_2 \nu(Q_2) \tag{4.2.34}$$

provided that $k \geq C\rho$.

Proof. We apply to u in the cylinder Q_2 Lemma 4.2.10 with the parameters

$$\mu_+ = 2^\beta k_1, \quad \omega = (2^\beta - \frac{1}{2}) k_1, \quad a = \frac{2^\beta - \frac{3}{4}}{2^\beta - \frac{1}{2}}.$$

It is clear that $k \geq C\rho$ implies $\omega \geq CR$. We obtain numbers ξ_1' and ξ_2' such that if

$$|\{(x,t) \in Q_2 : u(x,t) > k_1/2\}| \leq \xi_1' |Q_2| \tag{4.2.35}$$

and

$$\nu(\{(x,t) \in Q_2 : u(x,t) > k_1/2\}) \leq \xi_2' \nu(Q_2) \tag{4.2.36}$$

then

$$\operatorname*{ess\,sup}_{\frac{1}{2}Q_2} u(x,t) \leq \frac{3k_1}{4},$$

which leads to a contradiction. Hence, one of the conditions (4.2.35), (4.2.36) must be violated. Suppose that (4.2.35) is not true. Then using (4.2.8) we obtain (4.2.34). On the other hand, if (4.2.36) is not true we use (4.2.9) to get (4.2.33). □

Now we prove the 'concentration of positivity' lemma.

Lemma 4.2.18. *For any $\sigma \in (0,1)$ and $\lambda \in (0,1)$ there exist a point $(x',t') \in Q_2$ and a number $\eta = \eta(data, \beta, \theta, \lambda, \sigma) \in (0,1)$ such that the cylinder*

$$Q_4 = Q_4(\sigma, \lambda) = K^{x'}_{\eta R} \times [t' - \frac{\xi_1\theta_1}{8} h(x', \eta R) \left(\frac{\lambda k_1}{2}\right)^{2-p}, t'] \subset Q_2$$

and

$$|\{(x,t) \in Q_4(\sigma,\lambda) : u(x,t) > \lambda \frac{k_1}{2}\}| > \sigma |Q_4(\sigma,\lambda)|.$$

Proof. Pick a piecewise-smooth cut-off function ξ such that $0 \le \xi(x,t) \le 1$ on $2Q_2$, $|D\xi| \le \frac{4}{R}$, $|\xi_t| \le \frac{c}{\theta_1 h(x_0,R) k_1^{2-p}}$, ξ vanishes on the parabolic boundary of $2Q_2$ and $\xi(x,t) = 1$ on Q_2. Writing energy estimate (4.2.17) over the cylinder $2Q_2$ with $k = k_1/2$ and the cut-off function ξ we obtain

$$\iint_{Q_2} |D(u - \frac{k_1}{2})_-|^p \nu \, dxdt \tag{4.2.37}$$

$$\le \gamma \left(\left(1 + R^p + \frac{(CR)^p}{k_1^p}\right) \frac{k_1^p}{R^p} \nu(2Q_2) + \frac{k_1^2 |2Q_2|}{\theta_1 h(x_0,R) k_1^{2-p}} \right)$$

$$\le \frac{\gamma k_1^p}{R^p} \left(\nu(2Q_2) + \frac{R^p |2Q_2|}{\theta_1 h(x_0,R)} \right) \le \gamma \frac{k_1^p}{R^p} \nu(Q_2). \tag{4.2.38}$$

Applying the Hölder inequality we obtain from (4.2.38) that

$$\iint_{Q_2} |D(u - \frac{k_1}{2})_-| dxdt \le \gamma k_1 R^{-1} \nu(Q_2)^{1/p} \left(\iint_{Q_2} \nu^{\frac{1}{1-p}} dxdt \right)^{\frac{p-1}{p}}.$$

Since $\nu \in A_{1+\frac{p}{n}}$ implies $\nu \in A_p$, we finally arrive at

$$\iint_{Q_2} |D(u - \frac{k_1}{2})_-| dxdt \le \frac{\gamma k_1}{R} |Q_2|.$$

Now we change the variables as follows:

$$u = \frac{k_1}{2} w, \quad x - x_1 = Ry, \quad t - t_1 = k_1^{2-p} \theta_1 h(x_1, R) \tau.$$

In the new variables we obtain

$$\iint_{Q_5} |D(w-1)_-| dy d\tau \le \xi_3 |Q_5| = \xi_3,$$

where $Q_5 = K_1^0 \times [-1, 0]$ and $\xi_3 = \xi_3(data, \theta)$. From Lemma 4.2.17 we have

$$|\{(y,\tau) \in Q_5 : w(y,\tau) > 1\}| \ge \xi_1$$

with $\xi_1 = \xi_1(\beta, data, \theta)$. Let us show that there exists $\tau_* \in [-1, -\frac{\xi_1}{8}]$ such that simultaneously

$$\int_{K_1^0} |D(w-1)_-|(y, \tau_*) dy \le \frac{8\xi_3}{\xi_1}$$

and

$$|\{y \in K_1^0 : w(y,\tau_*) > 1\}| \geq \frac{\xi_1}{4}.$$

Denote

$$I(\tau) = \int_{K_1^0} |D(w-1)_-|(y,\tau)dy, \quad J(\tau) = |\{y \in K_1^0 : w(y,\tau) > 1\}|.$$

It is clear that

$$\left|\left\{\tau \in [-1,0] : I(\tau) > \frac{8\xi_3}{\xi_1}\right\}\right| < \frac{\xi_1}{8}. \tag{4.2.39}$$

Let us show that

$$\left|\left\{\tau \in [-1,0] : J(\tau) < \frac{\xi_1}{4}\right\}\right| < 1 - \frac{\xi_1}{4}. \tag{4.2.40}$$

Assume the converse. Then

$$\begin{aligned} &|\{(y,\tau) \in Q_5 : w(y,\tau) > 1\}| \\ &\leq \frac{\xi_1}{4} \cdot \left|\left\{\tau \in [-1,0] : J(\tau) < \frac{\xi_1}{4}\right\}\right| + 1 \cdot \left|\left\{\tau \in [-1,0] : J(\tau) \geq \frac{\xi_1}{4}\right\}\right| \\ &\leq \frac{\xi_1}{4} + \frac{\xi_1}{4} = \frac{\xi_1}{2}. \end{aligned}$$

From (4.2.39),(4.2.40) it follows immediately that

$$\left|\left\{\tau \in [-1,0] : I(\tau) \leq \frac{8\xi_3}{\xi_1} \text{ and } J(\tau) \geq \frac{\xi_1}{4}\right\}\right| \geq \frac{\xi_1}{8}.$$

Now the existence of the required τ_* is obvious.

Denote $w_1(y) = (w-1)_-(y,\tau_*)$. In this notation we have

$$\int_{K_1^0} |Dw_1|dy \leq \frac{8\xi_3}{\xi_1} \quad \text{and} \quad |\{y \in K_1^0 : w_1(y) = 0\}| \geq \frac{\xi_1}{4}.$$

From the result of [27] it follows that for any $\bar{\lambda} \in (0,1)$ and $\bar{\sigma} \in (0,1)$ there exist $y_1 \in K_1^0$ and $\bar{\eta} \in (0,1)$ such that

$$|\{y \in K_{\bar{\eta}}^{y_1} : w_1(y) < \bar{\lambda}\}| > \bar{\sigma}|K_{\bar{\eta}}^{y_1}|.$$

Return now to the original coordinates. We have proved that there exists $\tau_* \in [-1, -\frac{\xi_1}{8}]$, $x_2 \in K_R^{x_0}$, and $\bar{\eta} \in (0,1)$ such that $K_{\bar{\eta}R}^{x_2} \subset K_R^{x_1}$ and

$$|\{x \in K_{\bar{\eta}R}^{x_2} : u(x, t_1 + \tau_*\theta_1 k_1^{2-p}h(x_1,R)) > (1-\bar{\lambda})\frac{k_1}{2}\}| > \bar{\sigma}|K_{\bar{\eta}R}^{x_2}|.$$

Let κ be a constant such that $h(x, \varepsilon^{1/\kappa}r) \leq c\varepsilon h(x,r)$ uniformly for all x, r, and $\varepsilon \in [0,1]$. Denote $t_* = t_1 + \tau_* k_1^{2-p}h(x_1,R)$ and

$$t_{*,1} = t_* + \frac{\xi_1\theta_1}{8}h\left(x_2, (1-\bar{\sigma})^{\frac{1}{\kappa}}\frac{\bar{\eta}R}{2}\right)k_1^{2-p}.$$

Consider the cylinder $Q_6 = K^{x_2}_{\bar\eta R} \times [t_*, t_{*,1}]$. Pick a piecewise-smooth cut-off function $\xi = \xi(x)$ such that $0 \le \xi(x) \le 1$ for $x \in K^{x_2}_{\bar\eta R}$, ξ vanishes on the boundary of $K^{x_2}_{\bar\eta R}$, $\xi(x) = 1$ for $x \in K^{x_2}_{\frac{1}{2}\bar\eta R}$, and $|D\xi| \le \frac{4}{\bar\eta R}$. It is clear that $k_1 \ge C\bar\eta R$. Now we use energy estimate (4.2.17) with cut-off function ξ and $k = (1-\bar\lambda)\frac{k_1}{2}$ to obtain

$$\max_{t_* \le \tau \le t_{*,1}} \int_{K^{x_2}_{\frac{1}{2}\bar\eta R}} \left(u - (1-\bar\lambda)\frac{k_1}{2}\right)_-^2 dx$$

$$\le \gamma \left(\left(\frac{k_1}{2}\right)^2 |K^{x_2}_{\bar\eta R}|(1-\bar\sigma)\right.$$

$$\left. + \left(\frac{k_1^p}{(\bar\eta R)^p} + k_1^p + C^p\right) \nu(K^{x_2}_{\bar\eta R}) h(x_2, (1-\bar\sigma)^{\frac{1}{\kappa}}\bar\eta R)\theta_1 k_1^{2-p}\right)$$

$$\le \gamma k_1^2 |K^{x_2}_{\bar\eta R}| \left((1-\bar\sigma) + \theta_1(1-\bar\sigma)\left(1 + R^p + \frac{(C\bar\eta R)^p}{k_1^p}\right)\right)$$

$$\le \gamma k_1^2 |K^{x_2}_{\bar\eta R}|(1+\theta_1)(1-\bar\sigma).$$

Further, we estimate the left-hand side of the last inequality from below as

$$\int_{K^{x_2}_{\frac{1}{2}\bar\eta R}} \left(u - (1-\bar\lambda)\frac{k_1}{2}\right)_-^2 (x,t)dx$$

$$\ge \bar\lambda^2 \left(\frac{k_1}{2}\right)^2 |\{x \in K^{x_2}_{\frac{1}{2}\bar\eta R} : u(x,t) \le (1-2\bar\lambda)\frac{k_1}{2}\}|.$$

Combining the estimates, we see that for all $t \in [t_*, t_{*,1}]$

$$|\{x \in K^{x_2}_{\frac{1}{2}\bar\eta R} : u(x,t) \le (1-2\bar\lambda)\frac{k_1}{2}\}| \le \hat\gamma |K^{x_2}_{\frac{1}{2}\bar\eta \bar\lambda^{-2} R}|(1-\bar\sigma)$$

with $\hat\gamma = \hat\gamma(data, \theta)$. Consequently, in the cylinder $Q_7 = K^{x_2}_{\frac{1}{2}\bar\eta R} \times [t_*, t_{*,1}]$ we have

$$|\{(x,t) \in Q_7 : u(x,t) \le (1-2\bar\lambda)\frac{k_1}{2}\}| \le \hat\gamma |Q_7|(1-\bar\sigma)\bar\lambda^{-2}.$$

Now we break the base of Q_7 into 2^{ln} nonintersecting (up to a set of measure zero) congruent dyadic cubes $K^{z_j}_{2^{-l-1}\bar\eta R}$, $j = 1, \ldots, 2^{ln}$. Choose l so large that

$$\left(\frac{\lambda}{2}\right)^{2-p} h(z_j, 2^{-l-1}\bar\eta R) \le h\left(x_1, (1-\bar\sigma)^{1/\kappa}\frac{\bar\eta R}{2}\right) \tag{4.2.41}$$

for all z_j. We can do this using the properties (4.2.8), (4.2.9), which imply

$$h(z_j, 2^{-(l+1)}\bar\eta R) \le c2^{-l\kappa} h\left(x_2, \frac{\bar\eta R}{2}\right) \le c2^{-l\kappa}(1-\bar\sigma)^{-\frac{n+p}{\kappa}} h\left(x_2, (1-\bar\sigma)^{1/\kappa}\frac{\bar\eta R}{2}\right).$$

Now it is easy to see that (4.2.41) is satisfied if

$$l > \frac{n+p}{\kappa^2}\log_2\frac{1}{1-\bar{\sigma}} + \frac{1}{\kappa}\log_2 c + \frac{p-2}{\kappa}\log_2\frac{2}{\lambda}.$$

Denote

$$\tilde{Q}_j = K^{z_j}_{2^{-l-1}\bar{\eta}R} \times [t_*, t_{*,1}]$$

It is clear that for at least one j_* we have

$$|\{(x,t) \in \tilde{Q}_{j_*} : u(x,t) \le (1-2\bar{\lambda})\frac{k_1}{2}\}| \le \hat{\gamma}(1-\bar{\sigma})|\tilde{Q}_j|\bar{\lambda}^{-2}.$$

Let $l_1 \in \mathbb{N}$ be such that

$$2^{-l_1-1} \le \frac{(\lambda/2)^{2-p}h(z_{j_*}, 2^{-l-1}\bar{\eta}R)}{h(x_2, (1-\bar{\sigma})^{1/\kappa}\frac{\bar{\eta}R}{2})} \le 2^{-l_1}.$$

Denote

$$\xi_4 = 2^{-l_1-3}\xi_1\theta_1 k_1^{2-p}h(x_2, (1-\bar{\sigma})^{1/\kappa}\frac{\bar{\eta}R}{2}).$$

Break the cylinder $\tilde{Q}_{j_*}$ into the vertical layers

$$\tilde{Q}_{j_*,m} = K^{z_j}_{2^{-l-1}\bar{\eta}R} \times [t_* + (m-1)\xi_4, t_* + m\xi_4],$$

where $m = 1, 2, 3, \ldots, 2^{l_1}$. It is obvious that for at least one number m_* we have

$$|\{(x,t) \in \tilde{Q}_{j_*,m_*} : u(x,t) \le (1-2\bar{\lambda})\frac{k_1}{2}\}| \le \gamma(1-\bar{\sigma})\bar{\lambda}^{-2}|\tilde{Q}_{j_*,m_*}|.$$

Denote

$$t_{*,2} = t_* + (m_*-1)\xi_4.$$

Consider the cylinder

$$Q_8 = K^{z_{j_*}}_{2^{-l-1}\bar{\eta}R} \times [t_{*,2}, t_{*,2} + \frac{\theta_1\xi_1}{8}h(z_{j_*}, 2^{-l-1}\frac{\bar{\eta}R}{2})\left(\frac{\lambda k_1}{2}\right)^{2-p}].$$

In this cylinder we have

$$|\{(x,t) \in Q_8 : u(x,t) \le (1-2\bar{\lambda})\frac{k_1}{2}\}| \le 2\hat{\gamma}(1-\bar{\sigma})\bar{\lambda}^{-2}|Q_8|.$$

We conclude the proof by taking

$$\bar{\lambda} = \frac{1-\lambda}{2}, \quad \bar{\sigma} = 1 - \frac{1}{2\hat{\gamma}}\bar{\lambda}^2(1-\sigma).$$

$\square$

Take in the last lemma $\lambda = \frac{1}{2}$ and choose σ so large that in the cylinder

$$Q_8 = Q_4(\sigma, \frac{1}{4}) = K^{z'}_{\eta R} \times [t_2 - \frac{\xi_1\theta_1}{8}\left(\frac{k_1}{4}\right)^{2-p} h\left(z', \eta R\right),\, t_2]$$

the conditions of Lemma 4.2.11 are satisfied with

$$\mu_- = 0, \quad \omega = \frac{k_1}{4}, \quad a = \frac{1}{2}, \quad \theta = \frac{\xi_1\theta_1}{8}\omega^{2-p}.$$

We already have the estimate for the standard Lebesgue measure. Using relation (4.2.9) we can guarantee the smallness of $\nu(\{(x,t) \in Q_8 : u(x,t) \le \frac{k_1}{4}\})$. Denote $Q_9 = \frac{1}{2}Q_8$. Lemma 4.2.11 yields

$$\operatorname*{ess\,inf}_{Q_9} u(x,t) \ge \frac{k_1}{8} = \frac{1}{8}(1-\tau_0)^{-\beta}k.$$

Thus, we have found a point $z' \in K^{x_0}_\rho$, time t_2, and a number $\delta_1 = \delta_1(data, \beta) = \frac{\eta}{2}$ such that

$$\operatorname*{ess\,inf}_{K^{z'}_{\delta_1 R}} u(x,t_2) \ge \frac{1}{8}(1-\tau_0)^{-\beta}k.$$

Now we apply N times Lemma 4.2.16. We obtain the sequence $\hat{t}_j$, $j = 0, \ldots, N$ such that

$$\hat{t}_0 = t_2, \quad \hat{t}_j - \hat{t}_{j-1} = \gamma_1\left(\gamma_2^{j-1}\frac{1}{8}(1-\tau_0)^{-\beta}k\right)^{2-p} h(z', 2^{j-1}\delta_1 R),$$

and

$$\operatorname*{ess\,inf}_{K^{z'}_{2^j\delta_1 R}} u(x,\hat{t}_j) \ge \gamma_2^j\frac{1}{8}(1-\tau_0)^{-\beta}k$$

provided that

$$\operatorname*{ess\,inf}_{K^{z'}_{2^j\delta_1 R}} u(x,\hat{t}_j) \ge \gamma_3 C 2^j \delta_1 R \tag{4.2.42}$$

for all $j = 0, \ldots, N-1$. Choose the smallest N such that

$$2^N\delta_1\frac{1-\tau_0}{2}\rho \ge 3\rho.$$

Then $K^{x_0}_{2\rho} \in K^{z'}_{2^N\delta_1 R}$ and

$$2^N \le \frac{12}{(1-\tau_0)\delta_1} \Leftrightarrow N \le \log_2\frac{12}{(1-\tau_0)\delta_1}.$$

Hence,

$$\operatorname*{ess\,inf}_{K^{x_0}_{2\rho}} u(x,\hat{t}_N) \ge \frac{1}{8}\gamma_2^N(1-\tau_0)^{-\beta}k \ge \frac{1}{8}\left(\frac{12}{(1-\tau_0)\delta_1}\right)^{\log_2\gamma_2}(1-\tau_0)^{-\beta}k.$$

Choose $\beta = \log_2 \frac{1}{\gamma_2}$. (We can assume that $\gamma_2 < 1$). We obtain

$$\operatorname*{ess\,inf}_{K_\rho^0} u(x, \hat{t}_N) \geq \frac{1}{8}\left(\frac{12}{\delta_1}\right)^{\log_2 \gamma_2} k,$$

with $\gamma_2 = \gamma_2(data)$ and $\delta_1 = \delta_1(data, \theta)$. Clearly, the condition (4.2.42) is satisfied if

$$\frac{1}{8}\left(\frac{12}{\delta_1}\right)^{\log_2 \gamma_2} k \geq 6\gamma_3 C\rho,$$

which can be rewritten as

$$k \geq \Lambda_3' C\rho \quad \text{with} \quad \Lambda_3' = \Lambda_3'(data).$$

Note, that by our choice of parameters

$$(1-\tau_0)^{\beta(p-2)} \leq \left(12\delta_1^{-1}2^{-N}\right)^{\beta(p-2)} = \left(12\delta_1^{-1}\right)^{\beta(p-2)} \gamma_2^{N(p-2)}.$$

Hence, we estimate

$$\begin{aligned}\sum_{j=1}^{N}(\hat{t}_j - \hat{t}_{j-1}) &\leq \gamma_1 \frac{k^{2-p}}{8^{2-p}} h(x_0, 8\rho)\left(12\delta_1^{-1}\right)^{\beta(p-2)} \gamma_2^{N(p-2)} \sum_{j=1}^{N}(\gamma_2^{j-1})^{2-p} \\ &= \gamma_1 \frac{k^{2-p}}{8^{2-p}} h(x_0, 8\rho)\left(12\delta_1^{-1}\right)^{\beta(p-2)} \frac{1-\gamma_2^{N(p-2)}}{\gamma_2^{2-p}-1} \leq \Lambda_1 k^{2-p} h(x_0, \rho).\end{aligned}$$

Thus, we always have $\hat{t}_N \leq t_0 + \Lambda_1 k^{2-p} h(x_0, \rho)$ with $\Lambda_1 = \Lambda_1(data)$. Let

$$\Lambda_2' = \frac{1}{8}\left(\frac{12}{\delta_1}\right)^{\log_2 \gamma_2}.$$

Consider u in the cylinder $K_{2\rho}^{x_0} \times [\hat{t}_N, t_0 + \Lambda_1 k^{2-p} h(x_0, \rho)]$. Using Corollary 4.2.13 we can estimate

$$\operatorname*{ess\,inf}_{K_\rho^{x_0}} u(x, t_0 + \Lambda_0 k^{2-p} h(x_0, \rho)) \geq \frac{\Lambda_2' k}{2}\left(1 + \frac{t_0 + \Lambda_1 k^{2-p} h(x_0, \rho) - \hat{t}_N}{\delta(\Lambda_2' k)^{2-p} h(x_0, 2\rho)}\right)^{\frac{1}{2-p}}$$

if

$$\Lambda_2' k \geq C\rho \tag{4.2.43}$$

and

$$t_0 + \Lambda_1 k^{2-p} h(x_0, \rho) - \hat{t}_N \leq \delta(C\rho)^{2-p} h(x_0, 2\rho). \tag{4.2.44}$$

It is obvious that

$$t_0 + \Lambda_1 k^{2-p} h(x_0, \rho) - \hat{t}_N \leq (\Lambda_1 + \theta) k^{2-p} h(x_0, \rho).$$

Hence, inequalities (4.2.43) and (4.2.44) are satisfied if

$$k \geq \Lambda_3 C \rho$$

where

$$\Lambda_3 = \max\left(\Lambda_3', \frac{1}{\Lambda_2'}, \left(\frac{\Lambda_1+\theta}{\delta}\right)^{\frac{1}{p-2}}\right).$$

The theorem is proved with

$$\Lambda_2 = \frac{\Lambda_2'}{2}\left(1+\frac{\Lambda_1+\theta}{\delta}\right)^{\frac{1}{2-p}}.$$

□

Remark. It is obvious that the dependence on θ becomes critical when $\theta \to 0$. If $\theta \geq 1$ we can prove the theorem with $\theta = 1$.
Behaviour of constants as $p \to 2$. It is easy to see that the constants Γ_2 and Γ_3 in Theorem 4.2.1, obtained in the proof given here, deteriorate as p goes to 2. After the work we have done here it is relatively easy to prove, using the same type of argument as in [28], that the values of these constants in fact remain stable near $p = 2$.

Now we derive the (Hölder) continuity of solutions from Theorem 4.2.1.
Proof of Theorem 4.2.2. The proof is fairly standard and is in fact a modification of the proof of the Hölder continuity presented in [28].
Without loss, assume that

$$\operatorname*{ess\,sup}_{\Omega_T} |u| = M < \infty.$$

I. (*Preparation.*) Let $(x_0, t_0) \in \Omega \times (T_1, T_2]$. Consider the sequence of the cylinders

$$Q_j = K_{\rho_j}^{x_0} \times [t_0 - \omega_j^{2-p} h(\rho_j), t_0],$$

where we abbreviated $h(\rho_j) = h(x_0, \rho_j)$. The sequences $\{\omega_j\}_{j=0}^{\infty}$ and $\{\rho_j\}_{j=0}^{\infty}$ are defined by

$$\omega_j = \delta \omega_{j-1}, \quad \rho_j = \delta \rho_{j-1},$$

where $\delta \in (0,1)$ and $\varepsilon \in (0, \frac{1}{25})$. We assume that for all j the following inequality holds

$$3\delta^{2-p} h(\rho_{j+1}) \leq h(\rho_j).$$

which means that the height of Q_{j+1} is at least three times less than the height of Q_j. Denote

$$M_j = \operatorname*{ess\,sup}_{Q_j} u, \quad m_j = \operatorname*{ess\,inf}_{Q_j} u, \quad A_j = M_j - m_j,$$
$$t_j = t_0 - 2\omega_{j+1}^{2-p} h(\rho_{j+1}), \quad P_j = (x_0, t_j).$$

In the following we show that the constants δ, ε and Γ can be chosen such that if $A_0 \le 2\Lambda_1^{\frac{1}{p-2}}\omega_0$ and $\omega_0 \ge \Gamma C \rho_0$ then $A_j \le 2\Lambda_1^{\frac{1}{p-2}}\omega_j$ for all $j \in \mathbb{N}$.
In the cylinder Q_j define the following two functions:

$$u_{1,j} = M_j - u, \quad u_{2,j} = u - m_j.$$

It is obvious that $u_{1,j}$ and $u_{2,j}$ are nonnegative solutions of the equations of the same type as (4.2.1) with the same structural constants. Therefore the assertion of Theorem 4.2.1 holds for them with the same constants $\Lambda_1, \Lambda_2, \Lambda_3$. Denote

$$a_{1,j} = \lim_{s\to 0} \operatorname*{ess\,sup}_{Q_s^j} u_{1,j}, \quad a_{2,j} = \lim_{s\to 0} \operatorname*{ess\,sup}_{Q_s^j} u_{2,j},$$

where

$$Q_s^j = K_s^{x_0} \times [t_j - h(s), t_j].$$

It is clear that

$$a_{1,j} = M_j - \lim_{s\to 0} \operatorname*{ess\,inf}_{Q_s^j} u, \quad a_{2,j} = \lim_{s\to 0} \operatorname*{ess\,sup}_{Q_s^j} u - m_j.$$

Hence,

$$a_{1,j} + a_{2,j} \ge M_j - m_j = A_j.$$

Consequently, at least one of the numbers $a_{1,j}$, $a_{2,j}$ is no less than $\frac{A_j}{2}$. First, suppose that $a_{1,j} \ge \frac{A_j}{2}$. Then Theorem 4.2.1 yields

$$\operatorname*{ess\,inf}_{K_{\rho_{j+1}}^{x_0}} u_{1,j}\left(x, t_j + \Lambda_1 \left(\frac{A_j}{2}\right)^{2-p} h(\rho_{j+1})\right) \ge \Lambda_2 \frac{A_j}{2}$$

provided that

$$\frac{A_j}{2} \ge \Lambda_3 C \rho_{j+1}. \tag{4.2.45}$$

To obtain the lower bound on $u_{1,j}$ on the cylinder Q_{j+1} the waiting time must be sufficiently small, i.e.

$$\Lambda_1 \left(\frac{A_j}{2}\right)^{2-p} h(\rho_{j+1}) \le \omega_{j+1}^{2-p} h(\rho_{j+1}),$$

which is equivalent to

$$A_j \ge 2c_1\omega_{j+1}, \tag{4.2.46}$$

where we have denoted $c_1 = \Lambda_1^{\frac{1}{p-2}}$.

Further, choose τ_j such that the waiting time corresponding to $\tau_j \le a_{1,j}$, is equal to $2\omega_{j+1}^{2-p} h(\rho_{j+1})$:

$$\Lambda_1 \tau_j^{2-p} h(\rho_{j+1}) = 2\omega_{j+1}^{2-p} h(\rho_{j+1}),$$

whence

$$\tau_j = 2^{\frac{1}{2-p}} c_1 \omega_{j+1}.$$

Therefore, if (4.2.46) holds and

$$\tau_j = 2^{\frac{1}{2-p}} c_1 \omega_{j+1} \ge \Lambda_3 C \rho_{j+1} \tag{4.2.47}$$

we obtain that

$$\operatorname*{ess\,inf}_{Q_{j+1}} u_{1,j} \ge \Lambda_2 2^{\frac{1}{2-p}} c_1 \omega_{j+1}. \tag{4.2.48}$$

From (4.2.48) it easily follows that

$$A_{j+1} \le A_j - \Lambda_2 2^{\frac{1}{2-p}} c_1 \omega_{j+1}. \tag{4.2.49}$$

We arrive at the same estimate if we assume that $a_{2,j} \ge \frac{A_j}{2}$ and (4.2.46) and (4.2.47) hold.

Suppose now that the following inequality holds:

$$A_j \le \gamma \Lambda_2 2^{\frac{1}{2-p}} c_1 \omega_{j+1}. \tag{4.2.50}$$

Then (4.2.49) implies

$$A_{j+1} \le \left(1 - \frac{1}{\gamma}\right) A_j. \tag{4.2.51}$$

From now on we assume that for each j condition (4.2.47) is satisfied. It is easy to see that it is so if $\varepsilon \le \delta$ and $\omega_0 \ge \Gamma C \rho_0$, where $\Gamma = 2^{\frac{1}{p-2}} c_1 \Lambda_3 \varepsilon \delta^{-1}$.

Now we choose the parameters γ and δ such that if condition (4.2.50) is satisfied on the j-th step then it is satisfied on the $(j+1)$-th step. Let (4.2.46) hold. Then

$$\begin{aligned} A_{j+1} &\le \left(1 - \frac{1}{\gamma}\right) A_j \le \left(1 - \frac{1}{\gamma}\right) \gamma \Lambda_2 2^{\frac{1}{2-p}} c_1 \omega_{j+1} \\ &= \left(1 - \frac{1}{\gamma}\right) \delta^{-1} \gamma \Lambda_2 2^{\frac{1}{2-p}} c_1 \omega_{j+2} \le \gamma \Lambda_2 2^{\frac{1}{2-p}} c_1 \omega_{j+2} \end{aligned}$$

if

$$\left(1 - \frac{1}{\gamma}\right) \delta^{-1} \le 1. \tag{4.2.52}$$

On the other hand, if (4.2.46) does not hold, we obtain

$$A_{j+1} \le A_j \le 2c_1 \omega_{j+1} = 2c_1 \delta^{-1} \omega_{j+2} \le \gamma \Lambda_2 2^{\frac{1}{2-p}} c_1 \omega_{j+2}$$

if

$$2\delta^{-1} \leq \gamma \Lambda_2 2^{\frac{1}{2-p}}. \tag{4.2.53}$$

Take $\delta = 1 - \frac{1}{\gamma}$. Then γ and δ can be chosen as

$$\gamma = 1 + \frac{2^{\frac{p-1}{p-2}}}{\Lambda_2} \quad \text{and} \quad \delta = \frac{2^{\frac{p-1}{p-2}}}{2^{\frac{p-1}{p-2}} + \Lambda_2}.$$

It is clear that $\delta \in \left(\frac{1}{2}, 1\right)$. (Naturally, we can assume that $\Lambda_2 < 1$). Fix these values of γ and δ. We have proved that

$$A_j \leq \left(2^{\frac{1}{2-p}} \Lambda_2 + 2\right) c_1 \omega_{j+1} = 2 c_1 \omega_j$$

for all $j \in \mathbb{N}$ if **(1)** this inequality is true for $j = 0$, **(2)** $\omega_0 \geq \Gamma C \rho_0$, **(3)** $\varepsilon \leq \delta$.

We define ε as the maximal number from the interval $(0, \frac{1}{25}]$ such that $h(x, \varepsilon\rho) \leq \frac{\delta^{p-2}}{3} h(x, \rho)$ for all $x \in \mathbb{R}^n$ and $\rho > 0$. From (4.2.10) it follows that $\varepsilon > 0$. It is obvious that $\varepsilon < \delta$.

II. (*Proof of the Hölder continuity.*) Let $\Omega'_T = \Omega' \times [t_1, T_2]$ where $\Omega' \Subset \Omega$ and $t_1 > T_1$. Denote

$$d_x = \operatorname{dist}(\Omega', \partial\Omega), \quad d_t = t_1 - T_1.$$

It is clear that

$$\operatorname*{ess\,osc}_{\Omega_T} u \leq 2M.$$

Let $(x_0, t_0) \in \Omega'_T$. Consider the family of shrinking cylinders

$$Q_j = K_{\rho_j}^{x_0} \times [t_0 - \omega_j^{2-p} h(x_0, \rho_j), t_0],$$

where the sequences $\{\omega_j\}_{j=0}^{\infty}$ and $\{\rho_j\}_{j=0}^{\infty}$ are constructed as above. Let

$$A_j = \operatorname*{ess\,osc}_{Q_j} u.$$

Choose ω_0 and ρ_0 such that **(1)** $\rho_0 \leq d_x$, **(2)** $h(x_0, \rho_0)\omega_0^{2-p} \leq d_t$, **(3)** $\omega_0 \geq \Gamma C \rho$, **(4)** $A_0 \leq 2c_1\omega_0$. Take

$$\omega_0 = \max\left(\frac{M}{c_1}, \Gamma C \rho_0\right).$$

Then the second relation is satisfied if one of the following holds:

$$\Lambda_1 M^{2-p} h(x_0, \rho_0) \leq d_t, \tag{4.2.54}$$

$$\rho_0^2 \leq d_t (\Gamma C)^{p-2}. \tag{4.2.55}$$

Denote

$$H(x, s) = \max\{\rho : h(x, \rho) \leq s\}.$$

Set

$$\rho_0 = \min\left\{d_x, \max\left((\Gamma C)^{\frac{p-2}{2}}\sqrt{d_t}, H\left(x_0, \frac{d_t}{\Lambda_1 M^{2-p}}\right)\right)\right\}.$$

Denote

$$Q_{r,s}^{x_0,t_0} = K_r^{x_0} \times [t_0 - s, t_0].$$

Let

$$j_* = \max\{j : Q_{r,s}^{x_0,t_0} \subset Q_j\}.$$

Using (4.2.11) we can estimate the height of Q_j as

$$T_j = \omega_j^{2-p} h(x_0, \varepsilon^j \rho_0) \geq c\varepsilon^{(n+p)j}\delta^{(2-p)j}\omega_0^{2-p}.$$

Hence,

$$\operatorname*{ess\,osc}_{Q_{r,s}^{x_0,t_0}} u \leq 2c_1\omega_0\delta^{j_*} \leq 2c_1\omega_0\delta^{-1}\max\left[\left(\frac{s}{c\omega_0^{2-p}}\right)^{\alpha_1}, \left(\frac{r}{\rho_0}\right)^{\alpha_2}\right] := \varphi(x_0, t_0, r, s),$$

where

$$\alpha_1 = \frac{1}{\log_\delta \varepsilon^{n+p}\delta^{2-p}}, \quad \alpha_2 = \frac{1}{\log_\delta \varepsilon}.$$

It is obvious that

$$\varepsilon^{n+p}\delta^{2-p} = \left(\frac{\varepsilon}{\delta}\right)^p \varepsilon^n\delta^2 < 1,$$

whence $\alpha_1 > 0$.

For $t \in (T_2, 2T_2 - T_1)$ and $x \in \Omega$ define $u(x,t) = u(x, 2T_2 - t)$. For $t \in (T_1, T_2]$ and $x \in \Omega$ denote $Q_s^{x,t} = K_s^x \times [t - s, t + s]$ and

$$\hat{u}(x,t) = \lim_{s\to 0}\frac{1}{|Q_s^{x,t}|}\int_{Q_s^{x,t}} u(y,\tau)\,dy\,d\tau.$$

Our estimates of $\operatorname{ess\,osc} u$ imply that this limit exists for all $(x,t) \in \Omega \times (T_1, T_2]$. By the Lebesgue-Besicovitch theorem $\hat{u} = u$ a.e. in $\Omega \times (T_1, T_2]$. It is easy to see that for any point $(x_0, t_0) \in \Omega'_T$ we have

$$\operatorname*{osc}_{\tilde{Q}_{r,s}^{x_0,t_0}} \hat{u} \leq \varphi(x_0, t_0, r, s),$$

where $\tilde{Q}_{r,s}^{x_0,t_0} = K_r^{x_0} \times (t_0 - s, t_0)$ if $t_0 < T_2$ and $\tilde{Q}_{r,s}^{x_0,t_0} = K_r^{x_0} \times (t_0 - s, t_0]$ if $t_0 = T_2$. $\square$

Proof of Theorem 4.2.3. We can assume without loss of generality that $(x_0, t_0) = (0, 0)$. We prove the *sup* estimate. The proof of the *inf* estimate is merely a repetition.

Denote $h(0,\rho)=h(\rho)$. For $j=0,1,2,\dots$ denote

$$\sigma_j=\sigma+\frac{1-\sigma}{2^j},\quad \tilde{\sigma}_j=\sigma+\frac{3}{2}\frac{1-\sigma}{2^j},\quad t_j=-\sigma_j\theta h(\rho),\quad \tilde{t}_j=-\tilde{\sigma}_j\theta h(\rho).$$

It is easy to see that

$$\frac{\sigma_j}{\sigma_{j+1}}=2\frac{1+(2^j-1)\sigma}{1+(2^{j+1}-1)\sigma}\le 2.$$

Consider the cylinders

$$Q_j=K_{\sigma_j\rho}\times[t_j,0]\quad\text{and}\quad \tilde{Q}_j=K_{\tilde{\sigma}_j\rho}\times[\tilde{t}_j,0].$$

Observe that, in view of (4.2.6),

$$\frac{\nu(Q_j)}{\nu(Q_{j+1})}\le C_\nu\left(\frac{\sigma_j}{\sigma_{j+1}}\right)^{n+p}\le C_\nu 2^{n+p}.$$

Analogously, using (4.2.11), we obtain

$$\frac{|Q_j|}{|Q_{j+1}|}\le\left(\frac{\sigma_j}{\sigma_{j+1}}\right)^n\frac{h(\sigma_j)}{h(\sigma_{j+1})}\le c\left(\frac{\sigma_j}{\sigma_{j+1}}\right)^{2n+p}\le c2^{2n+p}.$$

Introduce the sequences of piecewise-smooth cut-off functions $\{\phi_j(x)\}$, $\{\psi_j(t)\}$, $\{\bar{\phi}_j(x)\}$, $\{\bar{\psi}_j\}$ such that

1. $\phi_j(x)=1$ for $x\in K_{\sigma_j\rho}$, $\phi_j(x)=0$ outside $K_{\tilde{\sigma}_j\rho}$, $|D\phi_j(x)|\le\frac{6}{(1-\sigma)\rho}2^j$;
2. $\bar{\phi}_j(x)=1$ for $x\in K_{\tilde{\sigma}_{j+1}\rho}$, $\bar{\phi}_j(x)=0$ outside $K_{\sigma_j\rho}$, $|D\bar{\phi}_j(x)|\le\frac{6}{(1-\sigma)\rho}2^j$;
3. $\psi_j(t)=1$ for $t\ge t_j$, $\psi_j(t)=0$ for $t\le\tilde{t}_j$, $0\le(\psi_j)_t\le\frac{6}{(1-\sigma)\theta h(\rho)}2^j$;
4. $\bar{\psi}_j(t)=1$ for $t\ge-\tilde{t}_{j+1}$, $\bar{\psi}_j(t)=0$ for $t\le\tilde{t}_j$, $0\le(\bar{\psi}_j)_t\le\frac{6}{(1-\sigma)\theta h(\rho)}2^j$.

Denote $\xi_j(x,t)=\phi_j(x)\psi_j(t)$ and $\bar{\psi}_j(x,t)=\bar{\phi}_j(x)\bar{\psi}_j(t)$. For a number k, which will be specified later, denote

$$k_j=2k-\frac{1}{2^j}k.$$

Set

$$A_j=\frac{1}{\nu(Q_j)}\iint_{Q_j}(u-k_j)_+^p\nu\,dxdt,\quad B_j=\frac{1}{|Q_j|}\iint_{Q_j}(u-k_j)_+^2\,dxdt.$$

Note that for (x,t) such that $u(x,t)>k_{j+1}$ we also have

$$(u-k_j)_+>k_{j+1}-k_j=\frac{k}{2^{j+1}}.$$

It is easy to obtain the following estimate:

$$\max_{-\tilde{\sigma}_{j+1}\theta h(\rho)\le t\le 0}\int_{\tilde{K}_{j+1}}(u-k_{j+1})_+^2dx+C_0\iint_{\tilde{Q}_{j+1}}|D[(u-k_{j+1})_+\xi_{j+1}]|^p\nu dxd\tau$$
$$\le\gamma\left[\frac{2^j|Q_j|}{(1-\sigma)\theta h(\rho)}B_j+\frac{2^{pj}\nu(Q_j)}{(1-\sigma)^p\rho^p}A_j+\nu(Q_j)A_j+2^{pj}k^{-p}A_j\right]$$
$$\le\gamma\nu(Q_j)2^{pj}\left[\frac{1}{(1-\sigma)\theta h(\rho)}B_j+\left(\frac{1}{(1-\sigma)^p\rho^p}+1+k^{-p}\right)A_j\right].$$

To estimate the first term on the left-hand side we write energy inequality (4.2.16) with $\xi=\bar{\xi}_j$ over Q_j with $k=k_{j+1}$. To estimate the second term on the left-hand side we write energy inequality (4.2.16) with $\xi=\xi_{j+1}$ over the cylinder $\tilde{Q}_{j+1}$. Further, estimate

$$A_{j+1}$$
$$\le\left[\frac{1}{\nu(Q_{j+1})}\iint_{Q_{j+1}}(u-k_{j+1})_+^{ph}\nu dxd\tau\right]^{\frac{1}{h}}\left[\frac{1}{\nu(Q_{j+1})}\iint_{Q_{j+1}}\chi_{\{u>k_{j+1}\}}\nu dxd\tau\right]^{1-\frac{1}{h}}$$
$$\le\gamma 2^{p(j+1)(1-1/h)}\left[\frac{1}{\nu(\tilde{Q}_{j+1})}\iint_{\tilde{Q}_{j+1}}(u-k_{j+1})_+^{ph}\xi_{j+1}^{ph}\nu dxd\tau\right]^{\frac{1}{h}}\left[k^{-p}A_j\right]^{1-1/h}$$
$$\le\gamma 2^{p(j+1)(1-1/h)}[k^{-p}A_j]^{1-1/h}\left[\max_{\tilde{t}_{j+1}\le t\le 0}\left(\frac{1}{|\tilde{K}_{j+1}|}\int_{\tilde{K}_{j+1}}(u-k_{j+1})_+^2\xi_{j+1}^2dx\right)\right.$$
$$\left.\times\frac{\tilde{\rho}_{j+1}^p}{\nu(\tilde{Q}_{j+1})}\iint_{\tilde{Q}_{j+1}}|D[(u-k_{j+1})_+\xi_{j+1}]|^p\nu dxd\tau\right]^{1/h}$$
$$\le\gamma 2^{p(j+1)(1-1/h)}|\tilde{K}_{j+1}|^{\frac{p}{2}\left(\frac{1}{h}-1\right)}\frac{\tilde{\rho}_{j+1}^{p/h}}{\nu(\tilde{Q}_{j+1})^{1/h}}|\tilde{Q}_{j+1}|^{\frac{1}{h}\left[1+\frac{p}{2}(h-1)\right]}$$
$$\times 2^{\frac{pj}{h}\left[1+\frac{p}{2}(h-1)\right]}k^{p\left(\frac{1}{h}-1\right)}A_j^{1-1/h}$$
$$\times\left[(1-\sigma)^{-1}\rho^{-p}B_j+A_j((1-\sigma)^{-p}\rho^{-p}+1+k^{-p})\right]^{\frac{1}{h}\left[1+\frac{p}{2}(h-1)\right]}.$$

After obvious cancelations we obtain

$$A_{j+1}\le\gamma 2^{\gamma_4 j}k^{p\left(\frac{1}{h}-1\right)}\left[\frac{B_j}{1-\sigma}+\frac{A_j}{(1-\sigma)^p}+A_j\rho^p(1+k^{-p})\right]^{\frac{1}{h}\left(1+\frac{p}{2}(h-1)\right)}A_j^{1-1/h}. \tag{4.2.56}$$

Let $k>C\rho$. Denote

$$Z_j=B_j+A_j(1+\rho^p).$$

In this notation from (4.2.56) we obtain that

$$A_{j+1}\le\gamma 2^{\gamma_4 j}k^{p\left(\frac{1}{h}-1\right)}(1-\sigma)^{-\frac{p}{h}\left(1+\frac{p}{2}(h-1)\right)}Z_j^{\frac{1}{h}\left(1+\frac{p}{2}(h-1)\right)}A_j^{1-\frac{1}{h}}. \tag{4.2.57}$$

Arguing in the same way, we estimate

$$
\begin{aligned}
&B_{j+1} \\
&\le \left[\frac{1}{|Q_{j+1}|}\iint_{Q_{j+1}} (u-k_{j+1})_+^{ph_1}\,dx d\tau\right]^{\frac{2}{ph_1}} \left[\frac{1}{|Q_{j+1}|}\iint_{Q_{j+1}} \chi_{\{u>k_{j+1}\}} dx d\tau\right]^{1-\frac{2}{ph_1}} \\
&\le \gamma 2^{2(j+1)(1-\frac{2}{ph_1})} \left[\frac{1}{|\tilde{Q}_{j+1}|}\iint_{\tilde{Q}_{j+1}} (u-k_{j+1})^{ph_1}\xi_{j+1}^{ph_1}\,dx d\tau\right]^{\frac{2}{ph_1}} \left[k^{-2}B_j\right]^{1-\frac{2}{ph_1}} \\
&\le \gamma 2^{2(j+1)(1-\frac{2}{ph_1})} \left[\max_{\tilde{t}_{j+1}\le t\le 0}\left(\frac{1}{|\tilde{K}_{j+1}|}\int_{\tilde{K}_{j+1}} (u-k_{j+1})_+^2\xi_{j+1}^2\,dx\right)^{\frac{p}{2}(h_1-1)}\right. \\
&\quad \left.\times \frac{\tilde{\rho}_{j+1}^p}{\nu(\tilde{Q}_{j+1})}\iint_{\tilde{Q}_{j+1}} |D[(u-k_{j+1})_+\xi_{j+1}]|^p\nu dx\right]^{\frac{2}{ph_1}} \left[k^{-2}B_j\right]^{1-\frac{2}{ph_1}} \\
&\le \gamma 2^{\gamma_2 j} k^{2\left(\frac{2}{ph_1}-1\right)} Z_j^{\frac{2}{ph_1}+1-\frac{1}{h_1}} B_j^{1-\frac{2}{ph_1}} (1-\sigma)^{-\frac{2}{h_1}-p+\frac{p}{h_1}}.
\end{aligned}
$$

Furthermore,

$$
B_{j+1} \le \gamma 2^{\gamma_4 j} k^{2\left(\frac{2}{ph_1}-1\right)} Z_j^{1+\left(1-\frac{1}{h_1}\right)} (1-\sigma)^{-\frac{2}{h_1}-p+\frac{p}{h_1}} \tag{4.2.58}
$$

and

$$
(1+\rho^p)A_{j+1} \le (1+\rho^p)^{\frac{1}{h}} k^{p\left(\frac{1}{h}-1\right)} 2^{\gamma_4 j} Z_j^{1+\frac{p}{2}\left(1-\frac{1}{h}\right)} (1-\sigma)^{-\frac{p}{h}\left(1+\frac{p}{2}(h-1)\right)}. \tag{4.2.59}
$$

Choose now the numbers h and h_1 such that

$$
1-\frac{1}{h_1} = \frac{p}{2}\left(1-\frac{1}{h}\right).
$$

Summing inequalities (4.2.58) and (4.2.59) we obtain

$$
Z_{j+1} \le \left[(1+\rho^p)^{\frac{1}{h}} k^{p\left(\frac{1}{h}-1\right)} + k^{2\left(\frac{2}{ph_1}-1\right)}\right] \gamma 2^{\gamma_4 j} (1-\sigma)^{-\gamma_5} Z_j^{1+\left(1-\frac{1}{h_1}\right)},
$$

where

$$
\gamma_5 = \max\left(\frac{2-p}{h_1}+p, \frac{p}{h}+2\frac{2}{h_1}\right).
$$

By the hypergeometric convergence lemma $Z_j \to 0$ as $j \to \infty$ provided that

$$
\begin{aligned}
Z_0 &\le \gamma \left[(1+\rho^p)^{\frac{1}{h}} k^{2\left(\frac{1}{h_1}-1\right)} + k^{2\left(\frac{2}{ph_1}-1\right)}\right]^{\frac{h_1}{1-h_1}} (1-\sigma)^{\gamma_5\frac{h_1}{1-h_1}} \\
&= \gamma k^2 \left[(1+\rho^p)^{\frac{1}{h}} + k^{\left(\frac{4}{p}-2\right)\frac{1}{h_1}}\right]^{\frac{h_1}{1-h_1}} (1-\sigma)^{\gamma_5\frac{h_1}{1-h_1}}.
\end{aligned}
$$

Now assume that

$$k \geq (1+\rho^p)^{\frac{h_1 p}{h(4-2p)}} = (1+\rho^p)^{\frac{(p-2)h_1+2}{4-2p}}.$$

Then the last inequality holds if

$$Z_0 \leq \gamma k^2 (1+\rho^p)^{\frac{h_1}{h(1-h_1)}} (1-\sigma)^{\gamma_5 \frac{h_1}{1-h_1}}.$$

The last inequality is clearly satisfied if k is large enough. Thus, we get the estimate

$$\operatorname*{ess\,sup}_{Q_\sigma} u_+ \leq \gamma (1-\sigma)^{\gamma_5 \frac{h_1}{2(h_1-1)}}$$

$$\times \max\left(J(\rho), I(\rho) \left(\frac{1}{|Q|} \iint_Q |u|^2 dxdt \right)^{\frac{1}{2}} + I(\rho) \left(\frac{1}{\nu(Q)} \iint_Q |u|^p \nu dxdt \right)^{\frac{1}{2}} \right),$$

where

$$I(\rho) = (1+\rho^p)^{\frac{h_1}{2(h_1-1)}}$$

and

$$J(\rho) = \max\left(C\rho, (1+\rho^p)^{\frac{(p-2)h_1+2}{4-2p}} \right).$$

$\square$

Chapter 5

Growth Lemma and Harnack Inequality.

The aim of this chapter is to illustrate the 'Growth Lemma' ideology of E.M. Landis. The results in this chapter are classical. They are due to E.M. Landis, N.V. Krylov and M. Safonov (non-divergent case), J. Moser and E. De Giorgi (divergent case). This chapter was born as an intersection of the excellent review by V.A. Kondratiev and E.M. Landis, the lectures given by M. Safonov in Pavia, and the book of J. Malý and W.P. Ziemer. I try to give here the simplest proofs possible in each case and in some cases I also show the alternative ways.

Here I consider two types of equations simultaneously. The first type is the nondivergent equation

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u = 0, \tag{5.0.1}$$

where the matrix $\{a_{ij}\}$ is uniformly elliptic: there exist positive constants λ and Λ such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n.$$

A function u is called a solution (sub-, supersolution) to equation (5.0.1) in a domain Ω if it belongs to $W^{2,n}(\Omega)$ and satisfies (5.0.1) ($Lu \geq 0$, $Lu \leq 0$) a.e. in Ω.

The second type is the equation in divergent form,

$$\operatorname{div}(\mathbf{A}(x,u,\nabla u)) = 0, \tag{5.0.2}$$

where $\mathbf{A}$ is a Carathéodory function satisfying

$$\mathbf{A}(x,\xi)\cdot\xi \geq \lambda|\xi|^p \quad \text{for all} \quad \xi \in \mathbb{R}^n,$$
$$|\mathbf{A}(x,\xi)| \leq \Lambda|\xi|^{p-1},$$

where $p = const > 1$, and λ, Λ are positive constants. A function $u \in W^{1,p}(\Omega)$ is a solution(subsolution, supersolution) to equation (5.0.2) in a domain Ω if

$$\int_\Omega \mathbf{A}(x, \nabla u) \cdot \nabla \xi \, dx = 0 \quad (\leq 0, \ \geq 0)$$

for any test-function $\xi \in C_0^\infty(\Omega)$.

In both cases, the constant λ is called the lower ellipticity constant and the constant Λ is called the upper ellipticity constant. I shall also need their ratio

$$\nu = \frac{\Lambda}{\lambda}.$$

5.1 Non-divergent case.

First, we prove the Harnack inequality for solutions of nondivergent equation (5.0.1) to expose the essence of the proof. The divergent case is more technically involved and we leave it for later.

We start by formulating the Landis Growth Lemma.

Lemma 5.1.1. *Let u be a positive subsolution of equation* (5.0.1) *in a domain $D \subset B_{4R}(x_0)$ which has limit points on $\partial B_{4R}(x_0)$. Let $u = 0$ on $\partial D \cap B_{4R}(x_0)$. Denote $H = B_R(x_0) \setminus D$. There exists a function $\gamma : (0, 1] \to \mathbb{R}^+$ which is monotonically increasing, positive for positive values of τ and such that*

$$\sup_{D \cap B_{4R}(x_0)} u \geq \left(1 + \gamma\left(\frac{|H|}{|B_R|}\right)\right) \sup_{D \cap B_R(x_0)} u.$$

The proof of this lemma is based on the following auxiliary fact, which is often called 'The Growth Lemma in Thin Domains'. We follow the elementary proof given by M. Safonov.

Lemma 5.1.2. *Let u be a positive subsolution of equation* (5.0.1) *in a domain $D \subset B_{4R}(x_0)$ which has limit points on $\partial B_{4R}(x_0)$. Let $u = 0$ on $\partial D \cap B_{4R}(x_0)$. There exists a function $\gamma_1 : (0, 1] \to \mathbb{R}^+$ such that*

$$\sup_{D \cap B_R(x_0)} u \leq \gamma_1\left(\frac{|D|}{|B_{4R}|}\right) \sup_{D \cap B_{4R}(x_0)} u$$

and

$$\gamma_1(\tau) \to 0 \quad as \quad \tau \to 0.$$

Proof. Denote

$$M = \sup_{D \cap B_{4R}(x_0)} u.$$

Let $x_1 \in D \cap B_R(x_0)$. Consider the function

$$v(x) = u(x) - M\left(\frac{|x-x_1|}{3R}\right)^2.$$

It is clear that

$$(Lv)_- \le \frac{2M}{(3R)^2}\sum_{i=1}^n a_{ii}(x) \le \frac{2Mn\Lambda}{9R^2}$$

and

$$v \le 0 \quad \text{on} \quad \partial D \cap B_{4R}(x_0).$$

Applying now the Alexandrov's maximum principle to the function v in $D \cap B_{4R}(x_0)$, we obtain

$$\sup_{D\cap B_{4R}(x_0)} v_+ \le CR\left\|\frac{(Lv)_-}{\lambda}\right\|_{L^n(D)} \le C_2 M\left(\frac{|D|}{R^n}\right)^{1/n},$$

where $C_2 = C_2(n,\nu)$.

Since $u(x_1) = v(x_1)$ and x_1 is an arbitrary point from $B_R(x_0)$, the proof is completed with

$$\gamma_1(t) = C_2 t^{1/n}.$$

□

Before stating the next two lemmas, we note that the function $v(x) = |x - x_0|^{-s}$ is a subsolution of equation (5.0.1) for some positive constant $s = s(n,\nu)$ and any point x_0. Indeed, let $x_0 = (x_{0,1}, x_{0,2}, \dots, x_{0,n})$ and $\xi_i = (x_i - x_{0,i})/|x - x_0|$. The straightforward calculation shows that

$$\begin{aligned} &L|x-x_0|^{-s} \\ &= \left(s(s+2)\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j - s\sum_{i=1}^n a_{ii}(x)\right)|x-x_0|^{-s-2} \\ &\ge |x-x_0|^{-s-2} s\left((s+2)\lambda - n\Lambda\right) \ge 0 \end{aligned}$$

if

$$s > n\frac{\Lambda}{\lambda} - 2.$$

Lemma 5.1.3. *Let u be a positive subsolution of equation* (5.0.1) *in a domain $D \subset B_{4R}(x_0)$ which has limit points on $\partial B_{4R}(x_0)$. Let $u = 0$ on $\partial D \cap B_{4R}(x_0)$. Let $B_\rho(x') \subset B_R(x_0) \setminus D$. Then*

$$\sup_{D\cap B_{4R}(x_0)} u \ge \left(1 + \gamma_2\left(\frac{\rho}{R}\right)\right)\sup_{D\cap B_R(x_0)} u,$$

where $\gamma_2(t) > 0$ for $t > 0$ and it can be given by the formula

$$\gamma_2(t) = \frac{c_1 t^s}{1 - c_1 t^s}, \quad c_1 = c_1(n, \nu).$$

Proof. Take

$$v(x) = M \frac{1 + \left(\frac{\rho}{3R}\right)^{-s} - \left(\frac{|x-x'|}{3R}\right)^{-s}}{\left(\frac{\rho}{3R}\right)^{-s}}$$

where

$$M = \sup_{D \cap B_{4R}(x_0)} u.$$

It is clear that v is a supersolution in D. Since $v \geq 0$ on $\partial B_\rho(x')$, and $v(x)$ increases as the $|x - x'|$ increases, $v \geq 0$ everywhere in $\bar{D}$. In particular,

$$v \geq 0 \quad \text{on} \quad \partial D \cap B_{4R}(x_0).$$

Moreover, it is easy to see that

$$v \geq M \quad \text{on} \quad \partial B_{4R}(x_0)$$

since for $x \in \partial B_{4R}(x_0)$ we have $|x - x'| \geq 3R$, and consequently, $\left(\frac{|x-x'|}{3R}\right)^{-s} \leq 1$.

Using the comparison principle, we obtain that

$$u \leq v \quad \text{in} \quad D \cap B_{4R}(x_0).$$

Hence, for $x \in B_R(x_0)$ we have

$$\begin{aligned} u(x) \leq v(x) &\leq M \max_{B_R(x_0)} \left[1 - \frac{(3R/|x - x'|)^s - 1}{\left(\frac{\rho}{3R}\right)^{-s}}\right] \\ &= M \left[1 - c_1 \left(\frac{\rho}{R}\right)^s\right], \quad \text{with} \quad c_1 = 2^{-s} - 3^{-s}. \end{aligned}$$

□

An easy consequence of Lemma 5.1.3 is the following lemma.

Lemma 5.1.4. *Let v be a positive supersolution of equation* (5.0.1) *in a domain $D \subset B_{4R}(x_0)$. Let $v = 0$ on $\partial D \cap B_{4R}(x_0)$. Assume that there exists a ball $B_{\varepsilon R}(x') \subset B_R(x_0)$ such that $v \geq 1$ on $\partial B_{\varepsilon R}(x') \cap D$. Then*

$$v \geq c_1 \varepsilon^s \quad \textit{in} \quad B_R(x_0) \cap D.$$

Proof. Take $u = 1 - v$ and the domain

$$D_1 = D \setminus B_{\varepsilon R}(x') \cap \{u > 0\}.$$

The function u is a positive subsolution of equation (5.0.1) in D_1,

$$u = 0 \quad \text{on} \quad \partial D_1, \quad B_{\varepsilon R}(x') \subset B_R(x_0) \setminus D_1, \quad \text{and} \quad \sup_{D_1 \cap B_{4R}(x_0)} u = M \le 1.$$

Applying Lemma 5.1.3 we obtain that

$$\sup_{D_1 \cap B_R(x_0)} u \le M(1 - c_1 \varepsilon^s) \le 1 - c_1 \varepsilon^s.$$

Hence, for $x \in D$,

$$v(x) \ge \min(1, c_1 \varepsilon^s) = c_1 \varepsilon^s.$$

5.1.1 Proof of the Growth Lemma.

In the proof of the Growth Lemma, we use a statement which is a particular case of Lemma 5.1.2.

Lemma 5.1.5. *Let u be a positive subsolution of equation* (5.0.1) *in a domain $D \subset B_{4R}(x_0)$ which has limit points on $\partial B_{4R}(x_0)$. Let $u = 0$ on $\partial D \cap B_{4R}(x_0)$. There exists a positive constant $\delta = \delta(n, \nu)$ such that if*

$$|D \cap B_{4R}(x_0)| \le \delta |B_{4R}|$$

then

$$\sup_{D \cap B_{4R}(x_0)} u \ge 2 \sup_{D \cap B_R(x_0)} u$$

The following statement is an easy consequence of Lemma 5.1.2 — one just needs to apply this lemma to $u = 1 - v$.

Lemma 5.1.6. *Let v be a positive supersolution of equation* (5.0.1) *in a domain $D \subset B_{4R}(x_0)$ which has limit points on $\partial B_{4R}(x_0)$. Let $v = 1$ on $\partial D \cap B_{4R}(x_0)$. There exists a positive constant $\delta = \delta(n, \nu)$ such that if*

$$|D \cap B_{4R}(x_0)| \le \delta |B_{4R}|,$$

then

$$v \ge \frac{1}{2} \quad \text{in} \quad B_R(x_0) \cap D.$$

Step 1. Let

$$M = \sup_{D \cap B_{4R}(x_0)} u.$$

Consider the function

$$v = 1 - \frac{u}{M}.$$

It is clear that $0 \le v \le 1$ and v is a positive supersolution of equation (5.0.1). We shall prove that there exists a positive constant σ such that

$$v \ge \sigma \quad \text{on} \quad B_R(x_0),$$

where the constant σ depends only on n, ν and $\frac{|H|}{|B_R|}$.

Denote

$$D_1 = D \quad \text{and} \quad H_1 = B_R(x_0) \setminus D_1.$$

Pick a number h such that

$$|B_{(1-h)R}(x_0) \cap H_1| = \frac{|H_1|}{2}.$$

It is obvious that

$$h \ge \varphi\left(\frac{|H|}{|B_R|}\right) > 0,$$

where

$$\varphi(t) = 1 - \left(1 - \frac{t}{2}\right)^{1/n}.$$

Note that $0 < \phi(t) < 1 - 2^{-n}$ for $t \in (0,1)$. Denote

$$\tilde{H}_1 = B_{(1-h)R}(x_0) \cap H_1.$$

Let x be a density point of $\tilde{H}_1$. Let $r(x)$ be the greatest number from the interval $(0, h]$ such that

$$|B_{4r(x)}(x) \setminus \tilde{H}_1| \le \delta |B_{4r(x)}|$$

where δ is a number from Lemma 5.1.2. Since for a density point x

$$\frac{|B_{4r}(x) \setminus \tilde{H}_1|}{|B_{4r}|} \to 0 \quad \text{as} \quad r \to 0,$$

it is obvious that $r(x) > 0$. There are two separate cases:

1. For all density points of $\tilde{H}_1$ we have $r(x) < h$;
2. There exists a point x such that $r(x) = h$.

First, let the second alternative hold. We apply Lemma 5.1.6 to obtain $v \ge \frac{1}{2}$ in $B_h(x) = B_{r(x)}(x)$. Applying Lemma 5.1.4, we see that

$$v \ge \frac{1}{2} c_1 h^s \ge \frac{c_1}{2}\left(\varphi\left(\frac{|H|}{|B_R|}\right)\right)^s$$

in $D_1 \cap B_R$ which completes the proof.

Second, assume that the first alternative holds. Then for all density points of $\tilde{H}_1$ we have,

$$|B_{4r(x)}(x) \setminus H_1| = \delta |B_{4r(x)}|.$$

We immediately see that

$$v_1(x) = v(x) \geq \frac{1}{2} \quad \text{in} \quad B_{r(x)}(x) \cap D.$$

Denote

$$v_2 = 2v_1, \quad D_2 = \{x \in D \,:\, v_2 < 1\}, \quad H_2 = B_R(x_0) \setminus D_2.$$

Using the Vitali Lemma, we can find a family of disjoint balls $\{B_j^1\}_{j=1}^{\infty} = \{B_{r(x_j)}(x)\}_{j=1}^{\infty}$ such that

$$\tilde{H}_1 \subset \cup_{j=1}^{\infty} 3B_j^1 \quad \text{up to a set of measure zero,}$$

where $3B_j^1$ stands for the ball with the same center as B_j^1 and tripled radius. Hence,

$$\sum_{j=1}^{\infty} |B_j^1| \geq 3^{-n} |\tilde{H}_1| \geq \frac{1}{2 \cdot 3^n} |H_1|$$

and

$$\sum_{j=1}^{\infty} |B_j^1 \cap D_1| \geq \frac{\delta}{2 \cdot 3^n} |H_1|.$$

Consequently,

$$|\{x \in D_1 \cap B_R(x_0) \,:\, v_1(x) \geq \frac{1}{2}\}| \geq \frac{\delta}{2 \cdot 3^n} |H_1|,$$

whence

$$|H_2| \geq \left(1 + \frac{\delta}{2 \cdot 3^n}\right) |H_1|.$$

Now we apply to v_2 the same reasoning as we applied to v_1, replacing in our arguments D_1, H_1 and $\tilde{H}_1$ by D_2, H_2 and $\tilde{H}_2$, correspondingly. If there exists x — a density point of $\tilde{H}_2$, such that the second alternative holds, then

$$v_2(x) \geq \frac{c_1}{2} \left(\varphi\left(\frac{|H_2|}{|B_R|}\right)\right)^s \quad \text{in} \quad B_R(x_0) \cap D_2.$$

Since in $D \setminus D_2$ we have $v \geq \frac{1}{2}$, we obtain

$$v(x) \geq \frac{c_1}{4} \left(\varphi(|H|)\right)^s \quad \text{in} \quad D \cap B_R(x_0),$$

and the proof is completed.

If the second alternative holds, then we obtain

$$|\{x \in D_2 \cap B_R(x_0) \, : \, v_2(x) \geq \frac{1}{2}\}| \geq \frac{\delta}{2 \cdot 3^n} |H_2|.$$

Now let

$$v_k = 2^{k-1} v_1, \quad D_k = \{x \in D \cap B_R(x_0) \, : \, v_k < 1\}, \quad H_k = B_R(x_0) \setminus D_k.$$

Applying the same reasoning, we obtain that if for all $j = 1, \ldots, k-1$ the second alternative holds, and the sets $D_1, \ldots, D_{k-1}$ are non-empty, then

$$|H_k| \geq \left(1 + \frac{\delta}{2 \cdot 3^n}\right)^{k-1} |H_1|.$$

On the other hand, it is obvious that

$$|H_k| \leq |B_R|.$$

Consequently, for a number $k_0 \in \mathbb{N}$ such that

$$\left(1 + \frac{\delta}{2 \cdot 3^n}\right)^{k_0 - 1} |H| \geq |B_R(x_0)|$$

we have

$$v_{k_0 - 1} = 2^{k_0 - 2} v \geq 1 \quad \text{in} \quad D \cap B_R(x_0),$$

which implies

$$v \geq 2^{2 - k_0} \quad \text{in} \quad D \cap B_R(x_0).$$

If on a k^{th} step the second alternative holds, then the proof is also completed, with the constant

$$\sigma = \sigma_k = \frac{c_1}{2^k} \left(\varphi\left(\frac{|H|}{|B_R|}\right)\right)^s. \tag{5.1.1}$$

From the relations

$$\left(1 + \frac{\delta}{2 \cdot 3^n}\right)^{k_0 - 1} |H| \geq |B_R|,$$
$$\left(1 + \frac{\delta}{2 \cdot 3^n}\right)^{k_0 - 2} |H| < |B_R|$$

we obtain immediately that

$$2^{2 - k_0} \geq \left(\frac{|H|}{|B_R|}\right)^{1/\gamma_0}$$

where

$$\gamma_0 = \log_2 \left(1 + \frac{\delta}{2 \cdot 3^n}\right).$$

Since k ranges from 1 to k_0, the value of σ in inequality (5.1.1) is greater then or equal to σ_{k_0}, which completes the proof.

The constant σ is given by

$$\sigma = \frac{c_1}{4}\left(\frac{|H|}{|B_R|}\right)^{1/\gamma_0}\varphi\left(\frac{|H|}{|B_R|}\right).$$

□

Remark. The proof of the Growth Lemma given here is not entirely 'measure-theoretic'. It uses the comparison with subsolutions to 'expand the positivity' from a ball with a controlled radius to the unit ball. Slightly more sophisticated argument allows one to prove the Growth Lemma knowing a priori only that the 'Growth Lemma in thin domains' holds.

5.1.2 Proof of the Harnack inequality.

Once we have proved the Growth Lemma, the proof of the Harnack inequality easily follows. Formally, we give here the proof for the non-divergent case, but as the reader shall see the proof in the divergent case is the same. The proof is greatly simplified by the device of Krylov and Safonov.

Theorem 5.1.7. *Let u be a nonnegative solution of equation* (5.0.1) *in a ball $B_{5R}(x_0)$. Then there exists a constant C, which depends only on n and ν, such that $u(x) \le Cu(y)$ for all $x, y \in B_R(x_0)$*

Assume, without loss, that u is a positive solution of equation (5.0.1) in the ball $B_4(y)$. We shall prove that there exists a constant $C = C(n, \nu)$ such that $u(x) \ge Cu(y)$ for all $x \in B_1(y)$. It is easy to see that this fact implies the Harnack inequality. Indeed, if u is a nonnegative solution in the ball $B_5(0)$, then for $x \in B_1(0)$ we have $B_4(x) \in B_5(0)$ and $0 \in B_1(x)$. Thus, we can write the following chain of inequalities:

$$u(0) \ge Cu(x) \ge C^2u(0),$$

which is the full Harnack inequality.

Let $\beta > 0$ be a number to be defined later. On the interval $[0, 1)$ consider two functions

$$m(\tau) = (1-\tau)^{-\beta} \quad \text{and} \quad n(\tau) = \sup_{|x|<\tau} u(x).$$

Denote the maximal root of the equation $m(\tau) = n(\tau)$ by τ_0. Since $m(\tau) \to \infty$ as $\tau \to 1$ and $n(\tau)$ stays uniformly bounded, it is clear that $\tau_0 < 1$.

Let $x_1 \in B_{\tau_0}(0)$ be a point such that

$$u(x_1) = n(\tau_0) = (1-\tau_0)^{-\beta}.$$

Note, that the ball $\tilde{B} = B_{\frac{1-\tau_0}{2}}(x_1) \subset B_{\frac{1+\tau_0}{2}}$. Hence,

$$\sup_{x\in\tilde{B}} u(x) \le \left(1 - \frac{1+\tau_0}{2}\right)^{-\beta} = 2^\beta (1-\tau_0)^{-\beta} = 2^\beta u(x_1).$$

Now, consider the function

$$v(x) = u(x) - \frac{u(x_1)}{2}.$$

Let

$$D = \{x \in \tilde{B} \, : \, v(x) > 0\} \quad \text{and} \quad \tilde{B}_1 = \{|x - x_1| < \frac{1-\tau_0}{8}\}.$$

It is clear, that

$$v = 0 \quad \text{on} \quad \partial D \cap \tilde{B}, \text{ and} \quad \sup_D v \le \left(2^\beta - \frac{1}{2}\right) u(x_1).$$

Using Lemma 5.1.2, find a number $\varepsilon_0 = \varepsilon_0(n, \nu) > 0$ such that if

$$|D| < \varepsilon_0 |\tilde{B}| \tag{5.1.2}$$

then

$$\sup_{D\cap\tilde{B}_1} v \le \frac{1}{2^{\beta+2} - 2} \sup_D v.$$

Assume that, indeed, (5.1.2) is true. Then

$$v(x_1) \le \sup_{D\cap\tilde{B}_1} v \le \frac{1}{2^{\beta+2}-2}\left(2^\beta - \frac{1}{2}\right) u(x_1) = \frac{u(x_1)}{4},$$

and

$$u(x_1) = v(x_1) + \frac{u(x_1)}{2} \le \frac{3u(x_1)}{4},$$

which is a contradiction. Thus, we have proved that

$$|\{x \in \tilde{B} \, : \, v(x) > 0\}| \ge \varepsilon_0 |\tilde{B}|.$$

Now, consider the ball $\tilde{B}_2 = \{|x - x_1| < 2(1-\tau_0)\}$. Denote

$$w(x) = \frac{u(x_1)}{2} - u(x), \text{ and} \quad D_1 = \{x \in \tilde{B}_2 \, : \, w(x) > 0\}.$$

It is clear that

$$D = \tilde{B} \setminus \bar{D}_1, \quad |\tilde{B} \setminus D_1| \ge \varepsilon_0 |\tilde{B}|,$$

and

$$\sup_{\tilde{B}_2} w \leq \frac{u(x_1)}{2}.$$

Applying The Growth Lemma 5.1.1, we obtain

$$\sup_{\tilde{B}} w \leq \sup_{D_1 \cap \tilde{B}} w$$
$$\leq \frac{1}{1+\gamma(\varepsilon_0)} \sup_{D_1 \cap \tilde{B}_2} w \leq \frac{u(x_1)}{2(1+\gamma(\varepsilon_0))},$$

where γ is the function introduced in Lemma 5.1.1. Hence,

$$\inf_{\tilde{B}} u = \frac{u(x_1)}{2} - \sup_{\tilde{B}} w \geq \frac{\gamma(\varepsilon_0)}{2(1+\gamma(\varepsilon_0))} u(x_1) := K_1 u(x_1),$$

where K_1 depends only on n and ν. Now we only need to apply Lemma 5.1.4 to complete the proof. Indeed, this lemma yields immediately that

$$\inf_{B_1} u \geq c_1 \left(\frac{1-\tau_0}{2}\right)^s K_1 u(x_1)$$
$$= c_1 2^{-s} K_1 (1-\tau_0)^{s-\beta}.$$

Choosing $\beta = s$, we eliminate the unknown quantity τ_0, and obtain finally

$$\inf_{B_1} u \geq c_1 2^{-s} K_1.$$

$\square$

5.2 Divergent case.

While in the non-divergent case it was the Growth Lemma ideology in conjunction with the Alexandrov maximum principle which led to the Harnack inequality of Krylov and Safonov, the Harnack inequality for the divergent case was known for about twenty years prior to that. There are two most commonly used methods of proving Harnack inequality for equations with divergent structure.

The first is via the John-Nirenberg lemma which asserts that if a function $f \in BMO$ then the exponent of some multiple of f, $e^{\delta f}$, belongs to the Muckenhoupt class A_2, where δ depends on the dimension of the underlying space $\mathbb{R}^n$ and on the 'BMO-norm' of f. This is applied to the function $\log u - \frac{1}{|B_R|}\int \log u \, dx$ to glue together the upper bound of $\sup u$ by $\|u\|_{L^q(B_R)}$ with positive q and the lower bound for $\inf u$ by $\|u\|_{L^p(B_R)}$ with negative p.

This way was historically the first, and it was invented by J. Moser to prove the Harnack inequality for elliptic equations

Another way uses the Bombieri lemma, and is also due to J. Moser, who used it prove the Harnack inequality for parabolic equations.

The method of E.M. Landis is not so widely known. The original proof of the Harnack inequality for elliptic equations given in Landis's book [82] used some complicated geometric constructions, which were quite distinct from either Moser's technique (iterating integral inequalities for powers of functions) or De Giorgi's technique (iterating integrals of level cuts of functions). Moreover, it is not clear whether original Landis's method can be applied to nonlinear equations (say, of the p-Laplace type). Nevertheless, the major part of the Landis's arguments is very useful, since it allows one to dispense with such heavy tools as John-Nirenberg or Bombieri's lemma. The main task is to prove the Growth Lemma, which presents in fact the same level of difficulty as proving the Hölder continuity of solutions, and then use the general argument, similar to the one used in the previous section for the nondivergent case.

Here I give two different proofs of the Growth lemma, one via the Moser's method, another one via the De Giorgi's method. The second method is in some sense more robust since it uses only the fact that a function belongs to a certain 'De Giorgi class' while the first method uses the logarithmic estimates.

5.2.1 De Giorgi-style proof of the Growth Lemma.

Let Ω be a domain in $\mathbb{R}^n$ and $p \in (1, \infty$.We say that a function u belongs to the De Giorgi class $\mathbf{DG}^p_\pm(\Omega; C)$ if $u \in W^{1,p}(\Omega)$ and for any $k \in \mathbb{R}$ and nonnegative cut-off function $\xi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega |D[(u-k)_\pm \xi]|^p \, dx \leq C \int_\Omega (u-k)^p_\pm |D\xi|^p \, dx.$$

It is easy to verify that any solution u of equation (5.0.2) belongs to the De Giorgi classes $\mathbf{BG}^p_\pm(\Omega; C)$ with the constant $C = C(n, p, \nu)$ independent of u. Indeed, take the test function $\zeta = (u-k)_\pm \xi^p$ where $\xi \in C_0^\infty(\Omega)$. The function ζ is not from C_0^∞, but we can use the standard approximation argument to show that in fact it is possible to take test functions from $W_0^{1,p}(\Omega)$. We obtain

$$\int_\Omega \mathbf{A}(x, u, \nabla u) \cdot \left(\xi^p \nabla (u-k)_\pm + p\xi^{p-1}(u-k)_\pm \nabla \xi\right) \, dx = 0.$$

Since $\nabla(u-k)_\pm = \pm\nabla u$ on the set where $(u-k)_\pm > 0$ and is zero otherwise, we use the ellipticity condition and the Hölder inequality to estimate

$$\begin{aligned}
\lambda \int_\Omega |\nabla(u-k)_\pm|^p \xi^p\,dx &\le \int_\Omega \mathbf{A}(x,u,\nabla u)\cdot\nabla(u-k)_\pm \xi^p\,dx \\
&= -p\int_\Omega (u-k)_\pm \xi^{p-1}\mathbf{A}(x,u,\nabla u)\cdot\nabla\xi\,dx \\
&\le p\Lambda \int_\Omega (u-k)_\pm \xi^{p-1}|\nabla\xi|\cdot|\nabla(u-k)_\pm|^{p-1}\,dx \\
&\le p\Lambda\left(\int_\Omega (u-k)_\pm^p|\nabla\xi|^p\,dx\right)^{1/p}\left(\int_\Omega |\nabla(u-k)_\pm|^p\xi^p\,dx\right)^{(p-1)/p}.
\end{aligned}$$

After the obvious cancelation and raising the result to the power p we obtain

$$\int_\Omega |\nabla(u-k)_\pm|^p\xi^p\,dx \le \left(\frac{p\Lambda}{\lambda}\right)^p \int_\Omega (u-k)_\pm^p|\nabla\xi|^p\,dx.$$

Applying the Leibnitz formula, we obtain the desired result:

$$\begin{aligned}
\int_\Omega |\nabla[(u-k)_\pm\xi]|^p\,dx &\le C(n,p)\left(\int_\Omega |\nabla(u-k)_\pm|^p\xi^p + (u-k)_\pm^p|\nabla\xi|^p\,dx\right) \\
&\le C(n,p,\nu)\int_\Omega (u-k)_\pm^p|\nabla\xi|^p.
\end{aligned}$$

In the same manner, it is easy to verify that any subsolution of equation (5.0.2) belongs to the class $\mathbf{DG}_+^p(\Omega;C)$ and any supersolution belongs to the class $\mathbf{DG}_-^p(\Omega;C)$ with the constant $C = C(n,p,\nu)$.

Before formulating the Growth Lemma, I shall state and prove two classical estimates. The combination of the following two lemmas produces the Growth lemma almost immediately.

The first lemma is the classical supremum estimate.

Lemma 5.2.1. *Let $B = B_{2R}(x_0)$ and $u \in \mathbf{DG}_+^p(B;C)$. Let $l \in \mathbb{R}$. Then*

$$\operatorname*{ess\,sup}_{B_R(x_0)} u \le l + K_1\left(\frac{1}{|B|}\int_B (u-l)_+^p\,dx\right)^{1/p}\left(\frac{|\{x\in B\,:\,u(x)>l\}|}{|B|}\right)^{\frac{1}{n+p}}.$$

with the constant K_1 depending only on n, p and C.

Proof. For $j = 0,1,2,\dots$ denote

$$\rho_j = R + 2^{-j}R, \quad k_j = l + \omega - 2^{-j}\omega, \quad B_j = B_{\rho_j}(x_0).$$

where the constant ω will be chosen later. For $j = 0,1,2,\dots$ let ξ_j be a $C_0^\infty(B_j)$ function such that $\xi_j(x) = 1$ for $x \in B_{j+1}$ and $|D\xi_j| \le \frac{16n}{2^{-j-1}R}$. Applying the multiplicative Sobolev–Gagliardo–Nirenberg inequality (Theorem 2.3.1), we easily obtain

$$\begin{aligned}
&\int_{B_j} [(u-k_j)_+\xi_j]^{p+\frac{p^2}{n}}\, dx \\
&\le C_1 \left(\int_{B_j} (u-k_j)_+^p \xi_j^p \, dx \right)^{p/n} \int_{B_j} |D[(u-k_j)_+\xi_j]|^p \, dx \\
&\le C_2 (2^{-j}R)^{-p} \left(\int_{B_j} (u-k_j)_+^p \, dx \right)^{1+\frac{p}{n}} .
\end{aligned}$$

Now, applying the Hölder inequality, we see that

$$\begin{aligned}
\int_{B_{j+1}} (u-k_j)_+^p \, dx &\le \left(\int_{B_{j+1}} (u-k_j)_+^{p+\frac{p^2}{n}} \, dx \right)^{\frac{1}{1+p/n}} |\{u>k_j\}\cap B_{j+1}|^{\frac{p/n}{1+p/n}} \\
&\le \left(\int_{B_{j+1}} [(u-k_j)_+\xi_j]^{p+\frac{p^2}{n}} \, dx \right)^{\frac{1}{1+p/n}} |\{u>k_j\}\cap B_{j+1}|^{\frac{p/n}{1+p/n}} \\
&\le C_3 \int_{B_j} (u-k_j)_+^p \, dx \left(\frac{|\{u>k_j\}\cap B_{j+1}|}{R^n} \right)^{\frac{p}{n+p}} \\
&\le C_3 \int_{B_j} (u-k_j)_+^p \, dx \left(\frac{|\{u>k_j\}\cap B_j|}{R^n} \right)^{\frac{p}{n+p}} .
\end{aligned}$$

Now denote

$$I_j = \int_{B_j} (u-k_j)_+^p \, dx, \quad Z_j = \frac{|\{u>k_j\}\cap B_j|}{R^n}.$$

It is easy to see that

$$I_{j+1} \le \int_{B_{j+1}} (u-k_j)_+^p \, dx \le C_4 I_j Z_j^{\frac{p}{n+p}}.$$

On the other hand, since $u-k_j > 2^{-j-1}\omega$ on the set where $u>k_{j+1}$,

$$Z_{j+1} \le R^{-n}(2^{-j-1}\omega)^{-p} \int_{B_{j+1}} (u-k_j)_+^p \, dx \le C_5 2^{pj} \omega^{-p} R^{-n} I_j Z_j^{\frac{p}{n+p}}.$$

Combining the last two inequalities, we easily obtain the following:

$$X_{j+1} \le C_6 b^j \omega^{-\alpha p} R^{-n\alpha} (X_{j+1})^{1+\alpha},$$

where

$$X_j = I_j Z_j^{\frac{p}{n+p}}, \quad b = 2^{\frac{p^2}{n+p}}, \quad \alpha = \frac{p}{n+p}.$$

Now, the lemma on the fast geometric convergence (Lemma 2.3.5) yields that

$$\lim_{j\to\infty} X_j = 0$$

provided that

$$X_0 \le C_7 \omega^p R^n.$$

Therefore, if we choose

$$\omega \ge \omega_0 = C_8 \left(\int_B (u-l)_+^p \, dx \right)^{\frac{1}{p}} \left(\frac{|\{u > l\} \cap B|}{R^n} \right)^{\frac{1}{n+p}}$$

we obtain

$$\operatorname*{ess\,sup}_{B_R(x_0)} u \le l + \omega,$$

which is exactly the assertion of the lemma. □.

The second lemma is often called 'the telescopic argument'.

Lemma 5.2.2. *Let $B = B_{4R}(x_0)$ and $u \in \mathbf{DG}_+^p(B; C)$. Fix $l \in \mathbb{R}^n$. Let $M = \operatorname{ess\,sup}_B (u-l)_+ < \infty$. Assume that $|\{x \in B_{2R}(x_0) : u(x) < l\}| > \alpha |B_{2R}|$ where $\alpha = const > 0$. Then for any $\varepsilon > 0$ there exists $\omega > 0$ such that*

$$|\{(u-l)_+ > M(1-\omega)\} \cap B_{2R}(x_0)| < \varepsilon |B_{2R}|.$$

Moreover, ω depends only on $\alpha, \varepsilon, n, p, C$ and is independent of l, M and u.

Proof. For $j = 0, 1, 2, \ldots$ denote

$$k_j = l + M(1 - 2^{-j}), \quad A_j = \{u > k_j\} \cap B_{2R}(x_0).$$

Note that

$$k_{j+1} - k_j = 2^{-j-1} M = \frac{l + M - k_j}{2}.$$

Observe also that

$$|B_{2R}(x_0) \setminus A_j| \ge \alpha |B_{2R}|$$

for all $j = 0, 1, 2, \ldots$ and

$$A_0 \supset A_1 \supset A_2 \supset \ldots \supset A_j \supset A_{j+1} \ldots.$$

Successively applying the De Giorgi-Poincaré inequality (Theorem 2.3.3), the condition of the lemma, the Hölder inequality and the definition of the De Giorgi class, we obtain

$$(k_{j+1} - k_j)|A_{j+1}| \le \frac{C_1 R^{n+1}}{|B_{2R}(x_0) \setminus A_j|} \int_{A_j \setminus A_{j+1}} |Du| \, dx,$$

$$\frac{C_1 R}{\alpha} \left(\int_{A_{j+1} \setminus A_j} |D(u-k_j)_+|^p \, dx \right)^{1/p} |A_j \setminus A_{j+1}|^{1-1/p}$$

$$C_1 \alpha^{-1} (l + M - k_j) |B|^{1/p} \, |A_j \setminus A_{j+1}|^{1-1/p}.$$

Hence,

$$|A_{j+1}| \le C_2 |B|^{1/p} |A_j \setminus A_{j+1}|^{1-1/p}.$$

Consequently,

$$|A_{j+1}|^{\frac{1}{1-1/p}} \le C_2 |B|^{\frac{1/p}{1-1/p}} |A_j \setminus A_{j+1}|.$$

Summing the last inequality over $j = 0, 1, 2, \dots, j_0 - 1$ and estimating the left-hand side from below we obtain

$$j_0 |A_{j_0}|^{\frac{1}{1-1/p}} \le C_2 |B|^{\frac{1/p}{1-1/p}} |A_0 \setminus A_{j_0}| \le C_2 |B|^{\frac{1}{1-1/p}}.$$

Thus, we have proved that

$$|A_{j_0}| \le \left(\frac{C_3}{j_0}\right)^{1-1/p} |B_{2R}(x_0)|,$$

which is the assertion of the lemma. □

Lemma 5.2.3. *Let $B = B_{4R}(x_0)$ and $u \in \mathbf{DG}^p_+(B; C)$. Fix $l \in \mathbb{R}$ and denote $D = \{x \in B : u(x) > l\}$. Denote $H = B_R(x_0) \setminus D$. There exists a function $\gamma : (0,1) \to \mathbb{R}^+$, which is monotonically increasing and positive for positive values of the argument, such that*

$$\operatorname*{ess\,sup}_{D \cap B_{4R}(x_0)} (u - l)_+ \ge \left(1 + \gamma\left(\frac{|H|}{|B_R|}\right)\right) \operatorname*{ess\,sup}_{D \cap B_R(x_0)} (u - l)_+.$$

The function γ depends only on n, p and C.

Proof. Denote $M = \operatorname{ess\,sup}_{B_{4R}(x_0)} (u - l)_+$.

First, use Lemma 5.2.1 to find such ε that if for some $t \in \mathbb{R}$

$$|\{(u - t)_+ > 0\} \cap B_{2R}(x_0)| < \varepsilon |B_{2R}|$$

then

$$\operatorname*{ess\,sup}_{B_R(x_0)} (u - t)_+ < \frac{1}{2} \operatorname*{ess\,sup}_{B_{2R}(x_0)} (u - t)_+. \tag{5.2.1}$$

Now, use Lemma 5.2.2 to find $\omega > 0$ such that

$$|\{(u - l)_+ > M(1 - \omega)\} \cap B_{2R}(x_0)| < \varepsilon |B_{2R}(x_0)|.$$

Set $t = l + M(1 - \omega)$ in (5.2.1) to obtain

$$\begin{aligned}\operatorname*{ess\,sup}_{B_R(x_0)} (u - l)_+ &\le t - l + \operatorname*{ess\,sup}_{B_R(x_0)} (u - t)_+ \\ &\le M(1 - \omega) + \frac{M\omega}{2} = M\left(1 - \frac{\omega}{2}\right),\end{aligned}$$

which is the required result. □

The following statement is the 'growth lemma' for 'supersolutions'.

Lemma 5.2.4. *Let $B = B_{4R}(x_0)$ and $u \in \mathbf{DG}^p_-(B;C)$. Fix $l \in \mathbb{R}$ and denote $D = \{x \in B \,:\, u(x) > l\}$. Denote $H = B_R(x_0) \setminus D$.*

$$\operatorname*{ess\,sup}_{D\cap B_{4R}(x_0)} (u-l)_- \geq \left(1+\gamma\left(\frac{|H|}{|B_R|}\right)\right) \operatorname*{ess\,sup}_{D\cap B_R(x_0)} (u-l)_-.$$

The function γ is the same as in Lemma 5.2.3

To prove the Harnack inequality, we need to prove the divergent counterpart of Lemma 5.1.4

Lemma 5.2.5. *Let $u \in \mathbf{DG}^p_-(B;C)$ where $B = B_{17R}(x_0)$. Assume that u is nonnegative in $B_{4R}(x_0)$ and $u \geq l > 0$ in a ball $B_{\varepsilon R}(x') \subset B_R(x_0)$. Then there exist constants K_2 and $s > 0$ such that*

$$\operatorname*{ess\,inf}_{B_R(x_0)} u \geq K_2 \varepsilon^s l.$$

The constants K_2 and s depend only on n, p and C.

Proof. Let $\alpha = \gamma(2^{-n})$ where γ is a function from the statement of Lemma 5.2.3. For $j = 0, 1, 2, \dots$ denote

$$B_j = B_{2^j \varepsilon R}(x'), \quad t_j = (\frac{\alpha}{1+\alpha})^j l.$$

Since u is nonnegative, $\operatorname{ess\,sup}(u-t_j)_- \leq t_j$. Note, that

$$|B_1 \setminus \{(u-t_0)_- > 0\}| \geq |B_0| = 2^{-n}|B_1|.$$

First, we apply Lemma 5.2.4 in the ball B_3 with $l = t_0$ to obtain that

$$\operatorname*{ess\,sup}_{B_1}(u-t_0)_- \leq \frac{1}{1+\alpha} \operatorname*{ess\,sup}_{B_3}(u-t_0)_- \leq \frac{l}{1+\alpha}.$$

Thus,

$$u \geq l\left(1 - \frac{1}{1+\alpha}\right) = \frac{\alpha l}{1+\alpha} = t_1 \quad \text{in} \quad B_1.$$

Now, take $l = t_1$ in the statement of Lemma 5.2.4 and apply this lemma in the ball B_4 to obtain

$$\operatorname*{ess\,sup}_{B_2}(u-t_1)_- \leq \frac{1}{1+\alpha} \operatorname*{ess\,sup}_{B_4}(u-t_1)_- \leq \frac{t_1}{(1+\alpha)}.$$

Thus,

$$u \geq t_1\left(1 - \frac{1}{1+\alpha}\right) = \frac{\alpha t_1}{1+\alpha} = t_2 \quad \text{in} \quad B_2.$$

Analogously, on each step we obtain

$$\operatorname*{ess\,sup}_{B_j}(u-t_{j-1})_- \leq \frac{t_{j-1}}{1+\alpha}.$$

and

$$\operatorname*{ess\,inf}_{B_j} u \geq t_{j-1}\left(1 - \frac{1}{1+\alpha}\right) = t_j.$$

Take the smallest natural number j_0 such that $B_R(x_0) \subset B_{j_0}$. It is clear that j_0 must satisfy

$$2^{j_0}\varepsilon \geq 2, \quad 2^{j_0}\varepsilon \leq 4.$$

Hence,

$$\operatorname*{ess\,inf}_{B_{j_0}} u \geq \left(\frac{\alpha}{1+\alpha}\right)^{-j_0} l \geq \alpha \left(\frac{\varepsilon}{4}\right)^{\log_2 \frac{\alpha}{1+\alpha}}.$$

It is obvious that $B_{j_0+2} \subset B_{17R}(x_0)$, so all our steps are justified. □

Remark. It is easy to see that the above argument does not directly depend on the fact that u is from the De Giorgi class - the proof relies only on the fact that the Growth Lemma is valid for u. So, this argument can be applied in the non-divergent case as well.

Now, the proof of the Harnack inequality goes along the lines of the proof we gave in the non-divergent case. The only difference is that, since now we do not assume any a priori continuity of a solution, we choose the point x_1 such that

$$\operatorname*{ess\,sup}_{\tilde{B}\cap B_{\tau_0}(0)} u = m(\tau_0).$$

Of course, all supremums have to be replaced by essential supremums. If we do a small work and prove the Hölder continuity of solutions, we can take a continuous representative of u and simply repeat the proof in the non-divergent case, using Lemma 5.2.3 in place of Lemma 5.1.1 and Lemma 5.2.5 in place of Lemma 5.1.4.

5.2.2 Moser-style proof of the Growth lemma.

Here we prove the Growth Lemma in the following form.

Lemma 5.2.6. *Let u be a subsolution of equation* (5.0.2) *in a ball $B_{4R}(x_0) \subset \mathbb{R}^n$. Let $m \in \mathbb{R}$. Denote $H = B_R(x_0) \setminus \{u > m\}$ and $\alpha = \frac{|H|}{|B_R|}$. There exists a function $\gamma : (0,1] \to \mathbb{R}^+$, which is positive for positive values of its argument, such that*

$$\operatorname*{ess\,sup}_{B_{4R}(x_0)}(u-m)_+ \geq (1+\gamma(\alpha)) \operatorname*{ess\,sup}_{B_R(x_0)}(u-m)_+.$$

The next lemma is a reformulation of the Growth Lemma for supersolutions. Its proof is merely a repetition of the proof of Lemma 5.2.6.

Lemma 5.2.7. *Let u be a supersolution of equation (5.0.2) in a ball $B = B_{4R}(x_0) \subset \mathbb{R}^n$. Let $M \in \mathbb{R}$. Denote $H = B_R(x_0) \setminus \{u < m\}$ and $\alpha = \frac{|H|}{|B_R|}$. There exists a function $\gamma : (0,1] \to \mathbb{R}^+$, which is positive for positive values of its argument, such that*

$$\operatorname*{ess\,sup}_{B_{4R}(x_0)}(M-u)_+ \geq (1+\gamma(\alpha)) \operatorname*{ess\,sup}_{B_R(x_0)}(M-u)_+.$$

Step 1. We start with the following observation. Let $f : [m, M) \to \mathbb{R}$ be a piecewise C^2 nonnegative function, such that $f', f'' \geq 0$. Let $\xi \in C_0^\infty(B)$. Then

$$\int_B |\nabla f(u)|^p \xi^p \, dx \leq C \int_B (f(u))^p |\nabla \xi|^p \, dx \tag{5.2.2}$$

where the constant C depends only on p and ν. First, note that

$$\begin{aligned}\int_m^u (f'(t))^p \, dt &= (f'(t))^{p-1} f(t) \mid_m^u - \int_m^u f(t) f''(t) (f'(t))^{p-1} \, dt \\ &\leq (f'(t))^{p-1} f(t) \mid_m^u \leq (f'(u))^{p-1} f(u),\end{aligned} \tag{5.2.3}$$

where we use the Newton-Leibnitz formula and the positivity of f, f', f''.

Next, in the definition of a weak subsolution take the test function

$$\chi = \xi^p \int_m^u (f'(t))^p dt. \tag{5.2.4}$$

We obtain

$$\int_B \mathbf{A}(x, u, \nabla u) \cdot \left[\xi^p (f'(u))^p \nabla u + p \xi^{p-1} \int_m^u (f'(t))^p \, dt \nabla \xi \right] dx \leq 0.$$

Using the ellipticity condition, inequality (5.2.3) and the Young inequality we obtain

$$\begin{aligned}\lambda \int_B (f'(u))^p |\nabla u|^p \xi^p \, dx &\leq \Lambda p \int_B |\nabla u|^{p-1} \xi^{p-1} |\nabla \xi| \left(\int_m^u (f'(t))^p \, dt \right) dx \\ &\leq \Lambda p \int_B |\nabla u|^{p-1} \xi^{p-1} |\nabla \xi| (f'(u))^{p-1} f(u) \, dx \\ &\leq \frac{\lambda}{2} \int_B (f'(u))^p |\nabla u|^p \xi^p \, dx + C\lambda \int_B (f(u))^p |\nabla \xi|^p \, dx.\end{aligned}$$

After the obvious cancellation, we finally obtain the inequality

$$\int_B (f'(u))^p |\nabla u|^p \xi^p \, dx \leq C \int_B (f(u))^p |\nabla \xi|^p \, dx,$$

which is the required one.

From inequality (5.2.2) we easily obtain the standard Caccioppoli type inequality which we use later:

$$\int_B |\nabla[f(u)\xi]|^p\,dx \le C\int_B (f(u))^p|\nabla\xi|^p\,dx \tag{5.2.5}$$

where the constant C depends only on p, n and ν.

Remark. Formally speaking, here and below I do 'incorrect' steps because $f'(u)$ can be unbounded on $[m,M)$ (although it is bounded on any interval $[m,M-\varepsilon]$, $\varepsilon>0$). Thus, the left-hand side of (5.2.2) could be divergent, and taking the test-function ξ of the form (5.2.4) could be unjustified. To circumnavigate this obstacle, we can consider the family of functions ($\varepsilon>0$)

$$f_\varepsilon(t)=\begin{cases} f(t) & \text{for}\quad t<M-\varepsilon,\\ f(M-\varepsilon)+f'(M-\varepsilon)\,(t-(M-\varepsilon)) & \text{for}\quad M-\varepsilon\le t<M.\end{cases}$$

Taking f_ε instead of f in the above reasoning, we obtain (5.2.2) and (5.2.5) with f_ε. Sending ε to 0, we obtain the desired inequality for f.

Step 2. Let $f(u):[m,M)\to\mathbb{R}^+$ be a nonnegative piecewise C^2 function such that $f',f''>0$. We prove the following standard sup-estimate:

$$\operatorname*{ess\,sup}_{B_R(x_0)} f(u)\le C\left(\frac{1}{R^n}\int_{B_{2R}(x_0)}(f(u))^p\,dx\right)^{1/p} \tag{5.2.6}$$

where the constant C depends only on n, p and ν.

Let $\sigma=1+\frac{p}{n}$. Denote $R^j=2^{1-j}R$ and $B_j=B_{R_j}(x_0)$ for $j=0,1,2,\ldots$. Let ξ_j be a C_0^∞ cut-off function such that $\xi_j=1$ on B_{j+1}, $\xi_j=0$ outside B_j and $|\nabla\xi_j|\le C2^{j+1}R^{-1}$ with $C=C(n)$. Introduce the functions and

$$f_j(u)=(f(u))^{\sigma^j},\quad j=0,1,2,\ldots.$$

It is obvious that for all j we have $f_j,(f_j)',(f_j)''\ge 0$ on $[m,M)$.

Now we use the multiplicative Sobolev inequality and inequality (5.2.5) to estimate

$$\begin{aligned}\int_{B_{j+1}}(f_{j+1}(u))^p\,dx &= \int_{B_{j+1}}(f_j(u))^{p+p^2/n}\,dx\le\int_{B_j}(f_j)^{p+p^2/n}\xi_j^{1+p/n}\,dx\\ &\le C\left(\int_{B_j}(f_j(u)\xi_j)^p\,dx\right)^{p/n}\int_{B_j}|\nabla[f_j(u)\xi_j]|^p\,dx\\ &\le C2^{p(j+1)}R^{-p}\left(\int_{B_j}(f_j(u))^p\,dx\right)^{1+p/n}.\end{aligned}$$

Thus, for all $j = 1, 2, \dots$ we have

$$\frac{1}{R^n} \int_{B_{j+1}} (f(u))^{p\sigma^{j+1}} \, dx \le C 2^{p(j+1)} \left(\frac{1}{R^n} \int_{B_j} (f(u))^{p\sigma^j} \, dx \right)^{1+p/n}.$$

Let $k \in \mathbb{N}$. Applying successively the last inequality with $j = 0, 1, 2, \dots, k-1$ we obtain

$$\frac{1}{R^n} \int_{B_k} (f(u))^{p\sigma^k} \, dx \le C (2^p)^{r_k} \left(\frac{1}{R^n} \int_{B_0} (f(u))^p \, dx \right)^{\sigma^k}, \tag{5.2.7}$$

where

$$r_k = k + (k-1)\sigma + (k-2)\sigma^2 + \dots + 2\sigma^{k-2} + \sigma^{k-1}.$$

It is easy to see that

$$\begin{aligned} r_k &= \frac{\sigma^k - 1 + \sigma^{k-1} - 1 + \dots + \sigma - 1}{\sigma - 1} \\ &= \frac{\sigma \frac{\sigma^{k+1} - 1}{\sigma - 1} - k}{\sigma - 1} \le \sigma^k \frac{\sigma^2}{\sigma - 1}. \end{aligned}$$

Raising inequality (5.2.7) to the power $1/(p\sigma^k)$ we obtain

$$\begin{aligned} \left(\frac{1}{R^n} \int_{B_k} (f(u))^{p\sigma^k} \, dx \right)^{\frac{1}{p\sigma^k}} &\le C 2^{pr_k/\sigma_k} \left(\frac{1}{R^n} \int_{B_0} (f(u))^p \, dx \right)^{1/p} \\ &\le C \left(\frac{1}{R^n} \int_{B_0} (f(u))^p \, dx \right)^{1/p} \end{aligned}$$

in view of our estimate for r_k. From the last inequality it follows that

$$\operatorname*{ess\,sup}_{B_R(x_0)} f(u) \le \sup_{q>1} \left(\frac{1}{|B_R(x_0)|} \int_{B_R x_0} (f(u))^q \, dx \right)^{1/q} \le C \left(\frac{1}{R^n} \int_{B_0} (f(u))^p \, dx \right)^{1/p}.$$

Step 3. Let

$$M = m + \operatorname*{ess\,sup}_{B_{4R}(x_0)} (u - m)_+$$

and define

$$f(u) = \ln \frac{M - m}{M - m - (u - m)_+}.$$

It is easy to see that $f'(t), f''(t) > 0$ for $t \in [m, M)$ and $\{f(u) = 0\} = \{u \le m\}$. Let ξ be a smooth cut-off function ξ such that $\xi = 1$ on $B_{2R}(x_0)$, $\xi = 0$ outside $B_{4R}(x_0)$, $0 \le \xi \le 1$ and $|\nabla \xi| \le C(n) R^{-1}$. In the definition of a weak subsolution we take the test function

$$\chi = \left[(M - m - (u - m)_+)^{1-p} - (M - m)^{1-p} \right] \xi^p.$$

Note that χ is non-negative and $\chi = 0$ on the set $\{u \le m\}$.

Using the ellipticity condition and the Young inequality, we obtain

$$\begin{aligned}
\lambda(p-1) &\int_{B_{4R}(x_0)} |\nabla(u-m)_+|^p (M-m-(u-m)_+)^{-p} \xi^p \, dx \\
&\le \Lambda p \int_{B_{4R}(x_0)} |\nabla(u-m)_+|^{p-1} (M-m-(u-m)_+)^{1-p} \xi^{p-1} |\nabla \xi| \, dx \\
&\le \frac{\lambda(p-1)}{2} \int_{B_{4R}(x_0)} |\nabla(u-m)_+|^p (M-m-(u-m)_+)^{-p} \xi^p \, dx \\
&\qquad + C\lambda \int_{B_{4R}(x_0)} |\nabla \xi|^p \, dx.
\end{aligned}$$

After obvious cancelations, we arrive at

$$\begin{aligned}
&\int_{B_{4R}(x_0)} |\nabla(u-m)_+|^p (M-m-(u-m)_+)^{-p} \xi^p \, dx \\
&\qquad \le C \int_{B_{4R}(x_0)} |\nabla \xi|^p \, dx \le C R^{n-p}
\end{aligned}$$

with $C = C(n, p, \nu)$. Thus, we have obtained the following inequality:

$$\int_{B_{2R}(x_0)} |\nabla f(u)|^p \, dx \le C R^{n-p}$$

with the constant C independent of u. Applying the De Giorgi–Poincare inequality we obtain

$$\begin{aligned}
&\left(\frac{1}{R^n} \int_{B_{2R}(x_0)} (f(u))^p \, dx \right)^{1/p} \\
&\le C \frac{R^{n+1}}{|\{f(u) = 0\} \cap B_{2R}(x_0)|} \left(\frac{1}{R^n} \int_{B_{2R}(x_0)} |\nabla [f(u)]|^p \, dx \right)^{1/p} \\
&\le C \alpha^{-1}
\end{aligned}$$

with $C = C(n, p, \nu)$. Substituting this estimate in (5.2.6) we obtain

$$\operatorname*{ess\,sup}_{B_R(x_0)} f(u) \le C \alpha^{-1}.$$

Hence,

$$\operatorname*{ess\,sup}_{B_R(x_0)} (u-m)_+ \le (M-m)(1 - e^{-C/\alpha}),$$

which is the required result. □

5.2.3 Recovering the 'Growth Lemma in Thin Domains'.

The difference between the proofs in the divergent case and in non-divergent case is that in the former we obtained the Growth Lemma straightaway while in the latter we first proved the Growth Lemma in thin domains and then used some argument to obtain it in its full strength. In fact, once we are in possession of the Growth Lemma, we can prove an analogue of Lemma 5.1.2. Note that even in the non-divergent case we obtain better estimate for the function $\gamma(t)$ (compare with Lemma 5.1.2).

Lemma 5.2.8. *Let u be a subsolution in a ball $B_{4R}(x_0)$ and $m \in \mathbb{R}$. Let $D = \{x \in B_{4R}(x_0) : u(x) > m\}$. There exists a function $\gamma_2 : (0, 4^n) \to \mathbb{R}^+$ such that*

$$\operatorname*{ess\,sup}_{B_R(x_0)}(u-m)_+ \leq \gamma_2\left(\frac{|D|}{|B_R|}\right) \operatorname*{ess\,sup}_{B_{4R}(x_0)}(u-m)_+$$

and $\gamma_2(t) \to 0$ as $t \to 0$. Moreover, the function γ_2 can be given by the formula

$$\gamma_2(t) = \kappa^2 \cdot \kappa^{-3t^{-1/n}/8}$$

where $\kappa = 1 + \gamma(1/2)$ and the function γ is the same as in the statement of Lemma 5.2.6 or Lemma 5.2.3.

Remark. The same statement also holds in the non-divergent case, the function γ coming from Lemma 5.1.1 and the function u being a solution in D.

ProcProof. Denote

$$M = \operatorname*{ess\,sup}_{B_R(x_0)}(u-m)_+.$$

Let Let $d \in \mathbb{N}$ — we will choose it later. Let x_1 be a point in $B_R(x_0)$ such that

$$\operatorname*{ess\,sup}_{B_{R/d}(x_1)}(u-m)_+ = M.$$

If $B_{4R/d}(x_1) \subset B_{4R}(x_0)$ and $|D| \leq \frac{|B_{R/d}|}{2}$ we can apply the Growth Lemma in the ball $B_{4R/d}(x_1)$ to conclude that

$$\operatorname*{ess\,sup}_{B_{4R/d}(x_1)}(u-m)_+ \geq (1+\gamma(1/2))M = \kappa M.$$

Now, pick a point

$$x_2 \in B_{4R/d}(x_1) \subset B_{R+4R/d}(x_0)$$

such that

$$\operatorname*{ess\,sup}_{B_{R/d}(x_2)}(u-m)_+ \geq \kappa M.$$

If $B_{4R/d}(x_2) \subset B_{4R}(x_0)$ and $|D| \le \frac{|B_{R/d}|}{2}$ we can apply the Growth Lemma in the ball $B_{4R/d}(x_2)$ to obtain

$$\operatorname*{ess\,sup}_{B_{4R/d}(x_2)} (u-m)_+ \ge \kappa^2 M.$$

We can repeat this argument k times if 1) $|D| \le \frac{|B_{R/d}|}{2}$, 2) $R + k\frac{4R}{d} \le 4R$. On the k-th step we obtain a point

$$x_k \in B_{R+4R(k-1)/d}(x_0)$$

such that

$$\operatorname*{ess\,sup}_{B_{4R/d}(x_k)} (u-m)_+ \ge \kappa^k M.$$

From the second condition, $R + k\frac{4R}{d} \le 3R$, we easily obtain that $k \le \frac{3d}{4}$. Since k is integer, we can take

$$k = [3d/4] \ge 3d/4 - 1.$$

Now,

$$\operatorname*{ess\,sup}_{B_{4R}(x_0)}(u-m)_+ \ge \operatorname*{ess\,sup}_{B_{4R/d}(x_k)} (u-m)_+ \ge \kappa^k M \ge \kappa^{-1}\kappa^{3d/4} M. \tag{5.2.8}$$

From the first condition, $|D| \le \frac{|B_{4R/d}|}{2}$, we obtain

$$d^n \le \frac{1}{2|D|/|B_R|}$$

which gives

$$d = \left[\left(\frac{1}{2|D|/|B_R|} \right)^{1/n} \right] \ge \left(\frac{1}{2|D|/|B_R|} \right)^{1/n} - 1.$$

Using this evaluation of d in (5.2.8) we conclude that

$$\operatorname*{ess\,sup}_{B_{4R}(x_0)}(u-m)_+ \ge \frac{1}{\kappa^2} \kappa^{3\xi^{-1/n}/8} M$$

where $\xi = |D|/|B_R|$. □

5.2.4 The Hölder continuity of solutions.

With the power of the Growth Lemma at our hands, it is very easy to prove the Hölder continuity of solutions.

Theorem 5.2.9. *Let u be a solution to equation (5.0.2) in a domain Ω. Then, up to a modification on a set of measure zero, u is Hölder continuous in Ω. The Hölder exponent depends only on n, p and ν.*

Proof. **Step1.** As usual, on the first step we prove the reduction of oscillation of a solution in a smaller ball. Let u be a solution to equation (5.0.2) in a ball $B = B_{4R}(x_0)$. Let $\operatorname{ess\,sup}_B u = M$ and $\operatorname{ess\,inf}_B u = m$. Denote $\xi = (m + M)/2$ and $\omega = M - m$. It is clear, that either

$$|\{u \geq \xi\} \cap B_R(x_0)| \geq \frac{|B_R(x_0)|}{2} \tag{5.2.9}$$

or

$$|\{u \geq \xi\} \cap B_R(x_0)| \geq \frac{|B_R(x_0)|}{2} \tag{5.2.10}$$

In the first case, we apply the Growth Lemma for supersolutions to conclude that

$$\operatorname*{ess\,sup}_{B_R(x_0)} (u - \xi)_- \leq \frac{1}{1 + \gamma(1/2)} \operatorname*{ess\,sup}_{B_{4R}(x_0)} (u - \xi)_-.$$

Thus, if (5.2.9) holds we obtain that

$$\operatorname*{ess\,inf}_{B_R(x_0)} u \geq \xi - \frac{\omega}{2(1 + \gamma(1/2))}$$

and

$$\operatorname*{ess\,osc}_{B_R(x_0)} u \geq M - \operatorname*{ess\,inf}_{B_R(x_0)} u \leq \omega \left(\frac{1}{2} + \frac{1}{2(1 + \gamma(1/2))} \right).$$

In the second case, we apply the Growth Lemma for subsolutions to conclude that

$$\operatorname*{ess\,sup}_{B_R(x_0)} (u - \xi)_+ \leq \frac{1}{1 + \gamma(1/2)} \operatorname*{ess\,sup}_{B_{4R}(x_0)} (u - \xi)_+.$$

Thus, if (5.2.10) holds we obtain that

$$\operatorname*{ess\,sup}_{B_R(x_0)} u \leq \xi + \frac{\omega}{2(1 + \gamma(1/2))}$$

and

$$\operatorname*{ess\,osc}_{B_R(x_0)} u \geq \operatorname*{ess\,sup}_{B_R(x_0)} u - m \leq \omega \left(\frac{1}{2} + \frac{1}{2(1 + \gamma(1/2))} \right)$$

So, we proved the following: If u is a solution of equation(5.0.2) in a ball $B_{4R}(x_0)$ then

$$\operatorname*{ess\,osc}_{B_R(x_0)} u \leq \delta \operatorname*{ess\,osc}_{B_{4R}(x_0)} u$$

where

$$\delta = \frac{1}{2} + \frac{1}{2(1 + \gamma(1/2))} < 1.$$

Step 2. Let u be a solution of (5.0.2) in a domain Ω and let $\operatorname{ess\,sup}_\Omega |u| = K < \infty$. Denote the Lebesgue representative of u by $\hat{u}$:

$$\hat{u}(x) = \lim_{\rho \to 0} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} u(y)\, dy.$$

By the Lebesgue-Besicovitch theorem, $\hat{u}$ coincides with u almost everywhere in Ω. Let $x, y \in \Omega$. Denote

$$d = \max(dist(x, \partial\Omega), dist(y, \partial\Omega)), \quad \rho = |x - y|, \quad z = (x + y)/2.$$

Assume that $d \geq 2\rho$. Then $B_{d/2}(z) \subset \Omega$.

Consider the sequence of the balls $B_j = B_{d/(2\cdot 4^j)}(z)$. It is clear that

$$\operatorname*{ess\,osc}_{B_0(x_0)} u \leq 2K.$$

Applying the above argument we see that

$$\operatorname*{ess\,osc}_{B_j} u \leq \delta^j 2K.$$

Denote the greatest number j such that $x, y \in B_j$ by j_0. It is clear, that

$$2\rho \geq \frac{d}{3 \cdot 4^{j_0}} > \frac{\rho}{2}.$$

Hence, j_0 is the unique integer in the interval $\left[\log_4 \frac{d}{\rho} - 1, \log_4 \frac{d}{\rho}\right)$. Thus,

$$|\hat{u}(x) - \hat{u}(y)| \leq \operatorname*{ess\,osc}_{B_{j_0}} u \leq 2K\delta^{j_0}$$
$$\leq \frac{2K}{\delta}\delta^{\log_4(d/\rho)} = \frac{2K}{\delta}\left(\frac{\rho}{d}\right)^{\log_4(1/\delta)}.$$

Denote $\Omega_\varepsilon = \{x \in \Omega \,:\, dist(x, \partial\Omega) < \varepsilon\}$. Let $\alpha = \log_4 \frac{1}{\delta}$. If $x, y \in \Omega_\varepsilon$ and $|x - y| < \varepsilon/2$ then

$$|\hat{u}(x) - \hat{u}(y)| \leq \frac{2K}{\delta}\left(\frac{|x - y|}{\varepsilon}\right)^\alpha. \tag{5.2.11}$$

If $|x - y| \geq \varepsilon/2$, we have

$$|\hat{u}(x) - \hat{u}(y)| \leq 2K \leq 2K\left(\frac{|x - y|}{\varepsilon}\right)^\alpha 2^\alpha. \tag{5.2.12}$$

Thus, for $x, y \in \Omega_\varepsilon$ we have

$$|\hat{u}(x) - \hat{u}(y)| \leq CK\left(\frac{|x - y|}{\varepsilon}\right)^\alpha$$

where $C = 2^{1+\alpha} + 2\delta^{-1}$. Hence,

$$[\hat{u}]_{\alpha;\Omega_\varepsilon} = \sup_{x, y \in \Omega_\varepsilon, x \neq y} \frac{|\hat{u}(x) - \hat{u}(y)|}{|x - y|^\alpha} \leq C\varepsilon^{-\alpha} \operatorname*{ess\,sup}_\Omega |u|.$$

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